Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

ON STRATIFIED WATER WAVES WITH CRITICAL LAYERS AND CORIOLIS FORCES

Miles H. Wheeler

University of Vienna
Oskar-Morgenstern-Platz 1
1090 Wien, Austria

(Communicated by the associate editor name)

Abstract. We consider nonlinear traveling waves in a two-dimensional fluid subject to the effects of vorticity, stratification, and in-plane Coriolis forces. We first observe that the terms representing the Coriolis forces can be completely eliminated by a change of variables. This does not appear to be well-known, and helps to organize some of the existing literature.

Second we give a rigorous existence result for periodic waves in a two-layer system with a free surface and constant densities and vorticities in each layer, allowing for the presence of critical layers. We augment the problem with four physically-motivated constraints, and phrase our hypotheses directly in terms of the explicit dispersion relation for the problem. This approach smooths the way for further generalizations, some of which we briefly outline at the end of the paper.

Contents

1. Introduction 2
1.1. Governing equations 2
1.2. Previous results 4
1.3. Plan of the paper 5
2. Eliminating the Coriolis parameter 5
2.1. Traveling waves 5
2.2. Time-dependent waves 6
3. Existence theory 7
3.1. Formulation 8
3.2. Fredholm index 0 12
3.3. An abstract lemma 14
3.4. Application of the lemma 16
4. Generalizations and other parametrizations 19
4.1. Coriolis forces 19
4.2. Wave number as the bifurcation parameter 20
4.3. Non-constant vorticity 20
4.4. The Boussinesq limit 21
REFERENCES 22

2010 *Mathematics Subject Classification.* Primary: 76B15, 35Q35; Secondary: 35R35.
Key words and phrases. Water waves; Stratified water waves; Vorticity; Free boundary problems; Bifurcation.

1. **Introduction.** This paper concerns traveling waves in a two-dimensional inviscid and incompressible fluid lying above a flat bed. The fluid is divided into one or more layers separated by internal interfaces across which the pressure is continuous, and is bounded above either by a free surface held at constant (atmospheric) pressure or else by a rigid lid. While the velocity field is incompressible, the density of the fluid is allowed to vary continuously within each layer and discontinuously across the internal interfaces. Similarly we allow for nonzero vorticity in each layer as well as the existence of closed streamlines. Finally, we allow for Coriolis forces perpendicular to the fluid velocity. Such terms appear in non-traditional f-plane approximations at the equator [7].

Our first result (Proposition 2.1 below) is that traveling-wave solutions with Coriolis parameter $\Omega \neq 0$ can be naturally associated to solutions with $\Omega = 0$ and conversely. In this sense the two problems are mathematically equivalent, even if their physical interpretations are different. We were surprised not to find this remarked upon in recent work on waves with Coriolis forces. The basic idea is simple: By incompressibility, the Coriolis terms in the momentum equations are a gradient and so can be absorbed into the pressure. In general this redefinition of the pressure leaves forcing terms on the internal interfaces and free surface, but for traveling waves one can arrange for these forcing terms to vanish. The drawback is that the gravitational constant g must be replaced by $g - 2\Omega c$ where c is the wave speed. Branches of solutions with fixed g and variable c are therefore not preserved under this transformation.

Our second result (Theorem 3.1 below) is on the existence of periodic waves. We specialize to the two-layer case with a free surface, and require the vorticity and density to be constant in each layer. We also enforce four integral constraints which ensure that the average depths of the two layers are constant, that the wave speed c is physically defined, and that the average strength of the vortex sheet at the internal interface is zero. The results are stated entirely in terms of the formal dispersion relation $d(k, c) = 0$ between the wavenumber k and wave speed c of an infinitesimal wave. Especially since the linear operators involved are not Fourier multipliers, it is not immediately obvious that this should be possible. We state and prove a functional-analytic lemma which clarifies the issue and allows our existence result to be more easily generalized in a variety of directions.

1.1. **Governing equations.** Consider a configuration with $N \geq 1$ layers as in Figure 1. The layers are numbered $1, \dots, N$ starting with the deepest layer, while the internal interfaces are numbered $0, \dots, N$ with 0 corresponding to the flat bed and N to the free surface or rigid lid. Introducing a "reference thickness" $h_i > 0$ for each layer, the "reference height" of the ith interface is $h_0 + \cdots + h_i$, and we assume that the interface itself is a graph

$$S_i = \{z = h_0 + \cdots + h_i + \eta_i(x, t)\} \tag{1.1}$$

for some function η_i. On the flat bed $\eta_0 \equiv 0$. The ith layer is then

$$D_i = \{(x, z) : \eta_{i-1} < z - h_0 - \cdots - h_{i-1} < h_i + \eta_i\}, \tag{1.2}$$

where we are assuming $\eta_{i-1} < h_i + \eta_i$ so that the interfaces do not touch. Each layer has a velocity field (u_i, w_i), pressure field p_i, and density field $\rho_i > 0$, and we define the corresponding vortices by $\omega_i = u_{iz} - w_{ix}$. We will always work with classical solutions having at least the regularity $\eta_i \in C^1$ and $u_i, w_i, p_i, \rho_i \in C^1(\overline{D_i} \times \mathbb{R})$. For convenience we set $u_0 = w_0 = 0$.

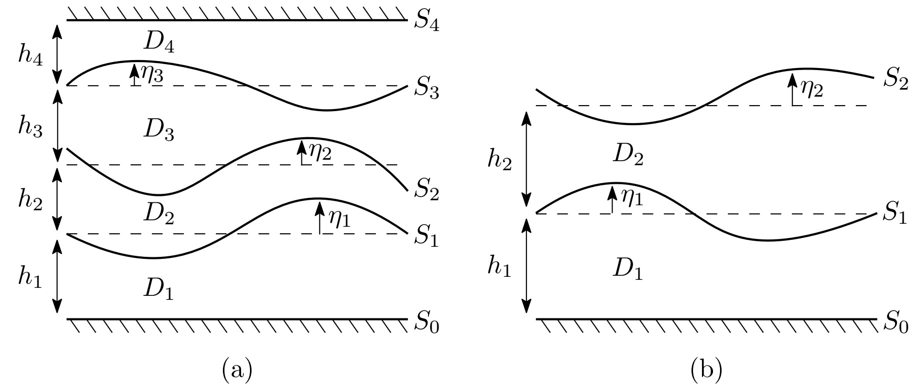

FIGURE 1. Fluid configurations with multiple layers using the notation (1.1) and (1.2). (a) A configuration with $N = 4$ layers and a rigid lid. (b) A configuration with $N = 2$ layers and a free surface. This is the type of configuration which will be considered in Section 3.

In each layer D_i the incompressible Euler equations

$$u_{it} + u_i u_{ix} + w_i u_{iz} + 2\Omega w_i = -p_{ix}/\rho_i, \tag{1.3a}$$

$$w_{it} + u_i w_{ix} + w_i w_{iz} - 2\Omega u_i = -p_{iz}/\rho_i - g, \tag{1.3b}$$

$$\rho_{it} + u_i \rho_{ix} + w_i \rho_{iz} = 0, \tag{1.3c}$$

$$u_{ix} + w_{iz} = 0 \tag{1.3d}$$

hold, where here g is the acceleration due to gravity and Ω is the angular velocity responsible for the Coriolis forces. At each interface S_i, including the bed S_0 and free surface S_N, there are kinematic boundary conditions

$$\eta_{it} - w_i + \eta_{ix} u_i = 0, \tag{1.3e}$$

$$\eta_{it} - w_{i+1} + \eta_{ix} u_{i+1} = 0, \tag{1.3f}$$

except that on S_N we have only (1.3e) and not (1.3f). These boundary conditions guarantee that fluid particles on S_i remain there for all time. The pressure is continuous across each internal interface,

$$p_i = p_{i+1} \text{ on } S_i, \quad i = 1, \dots, N-1. \tag{1.3g}$$

If the upper boundary S_N is a free surface then we have

$$p_N = p_{\text{atm}} \tag{1.3h}$$

there for some constant atmospheric pressure p_{atm}. If S_N is instead a rigid lid then we simply have

$$\eta_N = 0. \tag{1.3i}$$

By a traveling wave we mean a solution of (1.3) where the dependent variables $u_i, w_i, \rho_i, p_i, \eta_i$ depend on x and t only through the combination $x - ct$ for some wave speed $c \in \mathbb{R}$. Inserting this ansatz into (1.3), we are left with the time-independent problem

$$
\begin{aligned}
(u_i - c)u_{ix} + w_i u_{iz} + 2\Omega w_i &= -p_{ix}/\rho_i && \text{in } D_i, && i = 1,\dots,N, && (1.4a)\\
(u_i - c)w_{ix} + w_i w_{iz} - 2\Omega u_i &= -p_{iz}/\rho_i - g && \text{in } D_i, && i = 1,\dots,N, && (1.4b)\\
(u_i - c)\rho_{ix} + w_i \rho_{iz} &= 0 && \text{in } D_i, && i = 1,\dots,N, && (1.4c)\\
u_{ix} + w_{iz} &= 0 && \text{in } D_i, && i = 1,\dots,N, && (1.4d)\\
w_i - \eta_{ix}(u_i - c) &= 0 && \text{on } S_i, && i = 0,\dots,N, && (1.4e)\\
w_{i+1} - \eta_{ix}(u_{i+1} - c) &= 0 && \text{on } S_i, && i = 0,\dots,N, && (1.4f)\\
p_i &= p_{i+1} && \text{on } S_i, && i = 1,\dots,N-1, && (1.4g)\\
p_N &= p_{\text{atm}} && \text{on } S_N, && \text{(free surface case)}, && (1.4h)\\
\eta_N &= 0 && \text{on } S_N, && \text{(rigid lid case)}. && (1.4i)
\end{aligned}
$$

1.2. **Previous results.**

1.2.1. *Without Coriolis forces.* There is an extensive literature on solutions to (1.4) in the absence of Coriolis forces, even when we leave out important work on traveling-wave solutions to approximate models, on three-dimensional waves, and on the full time-dependent problem. We refer the reader to the surveys [15, 16, 32, 36, 37] and monograph [8] for a general overview. In terms of existence results for periodic waves, the simplest case of a single irrotational layer with a free surface dates back to Nekrasov [33] and Levi-Civita [28] in the 1920's. By comparison, the small-amplitude existence theory for waves with critical layers is not even a decade old [38] and the large-amplitude theory is quite recent [10].

Two-layer waves with vorticity and a rigid lid were constructed by Walsh, Bühler, and Shatah [40]. Like earlier work [11] with a single layer, they assume $u < c$ throughout the fluid which rules out the existence of critical layers. Matioc [30] has subsequently given an existence theory without this assumption. Compared to [40, 30], our existence result Theorem 3.1 treats the more complicated free-surface boundary condition. This introduces an additional unknown η_2 into the problem, and the dispersion relation (3.1) (for piecewise-constant vorticity) becomes quartic in the wave speed rather than quadratic. Perhaps more importantly, the dispersion relation loses monotonicity in the wavenumber k, so that there can be resonances between different wavenumbers k_1, k_2 for fixed c. On the other hand, while [40, 30] allow for general distributions of vorticity, we restrict to piecewise-constant vorticity. Our approach is not fundamentally restricted to this choice, however; see the remarks in Section 4.3.

We also mention a recent result of Wang [41], which treats rotational waves with general vorticity and a free surface boundary condition, but does not allow for critical layers. This paper constructs not only periodic waves but also waves which are solitary (localized) and "generalized solitary" (asymptotically periodic). Unlike [40, 30] which use the Crandall–Rabinowitz theorem [13] on bifurcation from a simple eigenvalue, Wang uses spatial dynamics techniques, in particular a center manifold theorem due to Mielke [31].

Our emphasis on the dispersion relation is similar in spirit to the work of Kozlov and Kuznetsov in [26] (also see [25]). They consider quite general rotational waves in a single constant-density layer, and treat two bifurcation problems: one where the Bernoulli constant is held fixed and the wavenumber k is varied, and another where the wavenumber k is fixed and the Bernoulli constant is varied. Our use of integral constraints is related to earlier work of Henry [18, 17] on constant-density rotational waves with constant depth and Walsh [39] on continuously stratified waves.

1.2.2. *With Coriolis forces.* Results on solutions of (1.4) with $\Omega \neq 0$ are fewer in number and comparatively recent. There has been work on explicit Gerstner-type solutions in Lagrangian coordinates [19, 22] as well as the existence of solutions via bifurcation theory [9, 29, 20]. There are also qualitative results on symmetry [21, 1] and particle trajectories [34, 24]. Several papers on Hamiltonian formulations of the time-dependent problem [4, 6, 5, 23] also include (rather formal) discussions of solitary traveling waves.

As we will show in Proposition 2.1 below, one can in fact always set $\Omega = 0$ in (1.4) after a simple change of variables. Thus the full strength of the classical theory for waves without Coriolis forces applies at once. In particular, some of the results for $\Omega \neq 0$ mentioned in the above paragraph can be directly inferred from earlier work with $\Omega = 0$.

1.3. **Plan of the paper.** In Section 2, we state and prove Proposition 2.1 on the equivalence between traveling waves with $\Omega \neq 0$ and $\Omega = 0$. For completeness we also briefly discuss a similar transformation for time-dependent waves which appears to be less useful. In Section 3, we prove our existence result Theorem 3.1. In Section 3.1, the full problem (1.4) is reduced to an abstract nonlinear equation in Banach spaces and the Crandall–Rabinowitz theorem is stated. In Section 3.2, we calculate the Fredholm indices of the relevant linear operators using standard techniques for elliptic problems. In Section 3.3, we give an abstract lemma which is useful for proving the remaining hypotheses of the Crandall–Rabinowitz theorem. In Section 3.4, we apply the lemma to complete the proof of Theorem 3.1. Finally, in Section 4 we consider several variants of Theorem 3.1, including a result where the wavenumber k is the bifurcation parameter. We have endeavored to write the paper in such a way that these variants and other generalizations are easily proved.

2. Eliminating the Coriolis parameter.

2.1. **Traveling waves.** In this section we show how the Coriolis terms involving Ω in the traveling-wave system (1.4) can be eliminated. The change of variables involves the "pseudo-stream functions" Ψ_i defined up to additive constants by

$$\Psi_{ix} = -\rho_i w_i, \quad \Psi_{iz} = \rho_i(u_i - c). \tag{2.1}$$

The existence of the Ψ_i follows immediately from the identity

$$(\rho_i(u_i - c))_x - (-\rho_i w_i)_z = -\rho_i(w_{iz} + u_{ix}) - (w_i\rho_{iz} + (u_i - c)\rho_{ix}) = 0,$$

which holds thanks to (1.4c) and (1.4d). The kinematic conditions (1.4e) and (1.4f) imply that Ψ_i is constant on S_i and S_{i+1}. Thus we can add constants to each of the Ψ_i to ensure that the normalization conditions

$$\begin{aligned} &\Psi_i = \Psi_{i+1} && \text{on } S_i, \quad i = 1, \dots, N-1, \\ &\Psi_N = 0 && \text{on } S_N \end{aligned} \tag{2.2}$$

are satisfied.

Proposition 2.1 (Eliminating Ω)**.** *The traveling-wave equations* (1.4) *are preserved under the transformation*

$$p_i \mapsto p_i' = p_i - 2\Omega\Psi_i, \qquad g \mapsto g' = g - 2\Omega c, \qquad \Omega \mapsto \Omega' = 0 \tag{2.3}$$

where here $\Psi_i = \Psi_i'$ *are defined by* (2.1)–(2.2)*.*

Proof. Subtracting the right hand side of (1.4a) from the left hand side, the terms involving p, g, Ω are

$$p_{ix}/\rho_i + 2\Omega w_i = (p_i' + 2\Omega\Psi_i)_x/\rho_i + 2\Omega w_i = p_{ix}'/\rho_i + 2\Omega' w_i.$$

Similarly, when we subtract the right hand side from the left hand side (1.4b), the relevant terms are

$$\begin{aligned} p_{iz}/\rho_i + g - 2\Omega u_i &= (p_i' + 2\Omega\Psi_i)_z/\rho_i + (g' + 2\Omega c) - 2\Omega u_i \\ &= p_{iz}'/\rho_i + 2\Omega(u_i - c) + g' - 2\Omega c - 2\Omega u_i \\ &= p_{iz}'/\rho_i + g' - 2\Omega' u_i. \end{aligned}$$

Since the equations (1.4c)–(1.4f) do not involve p, g, Ω, they are obviously preserved, and finally the dynamic boundary conditions (1.4g) are preserved thanks to the normalization (2.2). ☐

Remark 2.2. Proposition 2.1 continues to hold, with the same proof, when surface tension effects are included and also when (1.4) is generalized to allow for interfaces S_i that are not graphs.

Note that the transformation (2.3) leaves everything but the pressures p_i, gravitational constant g, and Coriolis parameter Ω unchanged. Thus the interfaces S_i, (pseudo-) stream functions such as Ψ_i, the trajectories of fluid particles, and the vorticities ω_i are all preserved.

2.2. **Time-dependent waves.** There does not appear to be an analogue of the transformation (2.3) which completely eliminates the Coriolis terms from (1.3). Under additional assumptions, one can, however, eliminate the Coriolis terms from the momentum equations (1.3a)–(1.3b) at the cost of adding forcing terms to the dynamic boundary conditions (1.3g) and (1.3h). While it is unclear if there are any applications, we outline such a transformation here for completeness.

Suppose, for instance, that the densities ρ_i are constant in each layer. Then by (1.3d) there exist stream functions Ψ_i in each layer satisfying $\Psi_{ix} = -\rho_i w_i$, $\Psi_{iz} = \rho_i u_i$ and unique up to an additive function of time t alone. The kinematic boundary conditions (1.3e)–(1.3f) imply that

$$\frac{1}{\rho_i}\frac{d}{dx}(\Psi_i|_{S_i}) = -w_i + u_i\eta_{i,x} = -\eta_{i,t} = \frac{1}{\rho_{i+1}}\frac{d}{dx}(\Psi_{i+1}|_{S_i})$$

on S_i for $i = 1, \ldots, N-1$, and so we can normalize the Ψ_i so that

$$\Psi_i = C_i(t) - \rho_i \int_0^x \eta_{it}(\tilde{x}, t)\, d\tilde{x}, \qquad \Psi_{i+1} = C_i(t) - \rho_{i+1} \int_0^x \eta_{it}(\tilde{x}, t)\, d\tilde{x}$$

on S_i for some functions $C_i(t)$. On the top S_N we can similarly arrange for

$$\Psi_N = \rho_N \int_0^x \eta_{N,t}(\tilde{x})\, d\tilde{x}.$$

With these normalizations in place, consider the transformation

$$p_i \mapsto p_i' = p_i - 2\Omega\Psi_i, \qquad g \mapsto g' = g, \qquad \Omega \mapsto \Omega' = 0,$$

where we are including $g \mapsto g$ merely to emphasize the difference with (2.3). As in the proof of Proposition 2.1, the momentum equations (1.3a)–(1.3b) are unchanged, as are (1.3d)–(1.3f). On the other hand, the boundary condition (1.3g) becomes

$$p_i' = p_{i+1}' + 2\Omega(\rho_i - \rho_{i+1}) \int_0^x \eta_{it}(\tilde{x}, t) d\tilde{x} \qquad \text{on } S_i.$$

For a rigid lid (1.3i) is unchanged, while the free surface boundary condition (1.3h) becomes

$$p_N' = p_{\text{atm}} + 2\Omega\rho_N \int_0^x \eta_{Nt}(\tilde{x}, t)\, d\tilde{x}.$$

3. **Existence theory.** This section is devoted to an existence result for (1.4). We take $N = 2$ layers with a free surface condition (see Figure 1b), and seek periodic waves with a fixed horizontal wave number $k = \kappa$. Abusing notation, we henceforth identify the interfaces S_0, S_1, S_2 and fluid layers D_1, D_2 with their intersections with a fundamental period $\{|x| < -\pi/\kappa\}$. We assume that the vorticities ω_1, ω_2 and densities ρ_1, ρ_2 in each layer are constants, and define the dimensionless ratio

$$r = \frac{\rho_1 - \rho_2}{\rho_2} > 0.$$

In light of Proposition 2.1 we set $\Omega = 0$ for simplicity, but see the discussion in Section 4.1. Introducing the shorthand

$$c_i = c - \omega_1 h_1 = \text{“relative wave speed at the interface”},$$
$$c_s = c - \omega_1 h_1 - \omega_2 h_2 = \text{“relative wave speed at the surface”},$$

the dispersion relation for this problem is then $d(k, c) = 0$ where

$$\begin{aligned} d(k,c) = \Big[\Big(& c_i^2 k \big((1+r) \coth kh_1 + \coth kh_2 \big) + c_i \big((1+r)\omega_1 - \omega_2 \big) - gr \Big) \times \\ & \Big(c_s^2 k \coth kh_2 + c_s \omega_2 - g \Big) \Big] - \Big(c_s c_i k \operatorname{csch} kh_2 \Big)^2. \end{aligned} \tag{3.1}$$

The formula (3.1) can of course be formally derived in many ways; it enters into our arguments in Section 3.4 as the determinant of a certain 6×6 matrix.

Theorem 3.1 (Existence of periodic waves)**.** *Fix $\kappa, h_1, h_2, r, \omega_1, \omega_2, g$ and set $\Omega = 0$. Suppose that at some speed c_* we have*

(i) (Simple root) *$d(\kappa, c_*) = 0$ and $d_c(\kappa, c_*) \neq 0$;*
(ii) (Non-resonance) *$d(\ell\kappa, c_*) \neq 0$ for $\ell \neq \pm 1, 0$; and*
(iii) (Non-critical surface and interface) *$c_* \neq \omega_1 h_2, \omega_1 h_1 + \omega_2 h_2$.*

Then there is an analytic curve of solutions to (1.4), parametrized by a small parameter ε, with the following properties.

(a) (Asymptotics) *As $\varepsilon \to 0$, we have the expansions*

$$\begin{aligned} \eta_1 &= \varepsilon \cos(\kappa x) + O(\varepsilon^2), \\ \eta_2 &= -\varepsilon \frac{c_s c_i \kappa \operatorname{csch} \kappa h_2}{c_s^2 \kappa \coth \kappa h_2 + c_s \omega_2 - g} \cos(\kappa x) + O(\varepsilon^2), \\ c &= c_* + O(\varepsilon^2). \end{aligned} \tag{3.2}$$

(b) (Average depths) *The layers have average depths h_1, h_2 in that*

$$\int_{S_1} \eta_1\, dx = \int_{S_2} \eta_2\, dx = 0. \tag{3.3}$$

(c) (Consistently-defined wave speed) *The wave speed c is uniquely determined by the requirement*

$$\int_{S_0} u_1 \, dx = 0. \tag{3.4}$$

(d) (Average vortex-sheet strength zero) *The net strength of the vortex sheet S_1 is zero in the sense that*

$$\int_{S_1} ((u_1, w_1) - (u_2, w_2)) \cdot (1, \eta_{1x}) \, dx = 0. \tag{3.5}$$

Before beginning the proof, let us comment on the integral conditions (3.3)–(3.5). While the constant depth condition (3.3) is certainly natural, many authors instead fix the volume fluxes M_1, M_2 defined in (3.8) below. This choice is not unreasonable from a physical point of view, and has some mathematical advantages. For further discussion we refer the reader to [18, 17]. Condition (2.3) is a normalization for the wave speed c, sometimes called "Stokes' first definition of the wave speed". It asserts that we are working in the unique reference frame where the horizontal velocity at the bed has average value zero. Many authors, for instance [11], instead fix c and use a Bernoulli constant such as B_2 in (3.6) below as the bifurcation parameter. Condition (3.5) at the internal interface is similar; it asserts that the *average* jump in tangential velocity is zero. This can be interpreted, for instance, as an effort to at least reduce the strength of the Kelvin–Helmholtz instability. An alternative would be to instead fix another Bernoulli constant, say B_1 in (3.6) below.

3.1. **Formulation.**

3.1.1. *Stream function formulation.* As in Section 2, we use incompressibility to introduce stream functions in each layer, except that we drop the prefactor ρ_i:

$$\Psi_{1x} = -w_1, \quad \Psi_{1z} = u_1 - c, \quad \Psi_{2x} = -w_2, \quad \Psi_{2z} = u_2 - c.$$

Using Bernoulli's law to eliminate the pressure, standard arguments lead to the following system:

$$\Delta\Psi_1 = \omega_1 \qquad \text{in } D_1, \tag{3.6a}$$
$$\Delta\Psi_2 = \omega_2 \qquad \text{in } D_2, \tag{3.6b}$$
$$\Psi_1 = M_1 \qquad \text{on } S_0, \tag{3.6c}$$
$$\Psi_1 = 0 \qquad \text{on } S_1, \tag{3.6d}$$
$$\Psi_2 = 0 \qquad \text{on } S_1, \tag{3.6e}$$
$$\Psi_2 = -M_2 \qquad \text{on } S_2, \tag{3.6f}$$
$$\tfrac{1}{2}|\nabla\Psi_2|^2 - (1+r)\tfrac{1}{2}|\nabla\Psi_1|^2 + gr\eta_1 = B_1 \qquad \text{on } S_1, \tag{3.6g}$$
$$\tfrac{1}{2}|\nabla\Psi_2|^2 + g\eta_2 = B_2 \qquad \text{on } S_2, \tag{3.6h}$$

with the constraints (3.3)–(3.5) becoming

$$\int_{S_1} \eta_1 \, dx = \int_{S_2} \eta_2 \, dx = 0, \tag{3.7a}$$

$$\int_{S_0} (\Psi_{1\zeta} + c) \, dx = 0, \tag{3.7b}$$

$$\int_{S_1} (\nabla\Psi_1 - \nabla\Psi_2) \cdot (1, \eta_{1x}) \, dx = 0. \tag{3.7c}$$

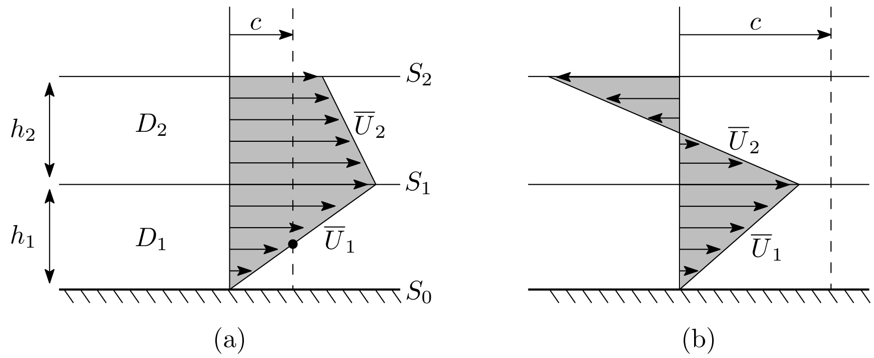

FIGURE 2. Shear flows $\overline{U}(z)$ corresponding to the stream functions $\overline{\Psi}_1, \overline{\Psi}_2$ in (3.9). Both flows have $\omega_2 < 0 < \omega_1$ and $c > 0$. (a) A flow with a critical layer at the marked point in D_1 where $\overline{U}_1 = c$. (b) A flow without a critical layer.

Here B_1, B_2 are Bernoulli constants, while M_1, M_2 are the x-independent volume fluxes in each layer,

$$M_1 = -\int_{-h_1}^{\eta_1} (u_1 - c)\, dz, \qquad M_2 = -\int_{\eta_1}^{h_2+\eta_2} (u_2 - c)\, dz. \tag{3.8}$$

Throughout the analysis we will hold $\omega_1, \omega_2, r, h_1, h_2, \kappa$ fixed, but allow M_1, M_2, B_1, B_2 and c to vary with the solution, c playing the role of bifurcation parameter. See Section 4 for related results with difference choices of parameters and constants.

3.1.2. *Trivial solutions.* We will perturb from the family of trivial (i.e. x-independent) solutions with $\eta_1, \eta_2 \equiv 0$ and

$$\begin{aligned} \Psi_1 = \overline{\Psi}_1(z;c) &:= (\omega_1 h_1 - c)z + \omega_1 \frac{z^2}{2}, \\ \Psi_2 = \overline{\Psi}_2(z;c) &:= (\omega_1 h_1 - c)z + \omega_2 \frac{z^2}{2}. \end{aligned} \tag{3.9}$$

These correspond to continuous piecewise-linear shear flows with horizontal velocity $\overline{U}_i = \Psi_{iz} + c$; see Figure 2. Inserting into (3.6) we discover formulas for M_1, M_2, B_1, B_2

$$M_1 = \overline{M}_1(c), \quad M_2 = \overline{M}_2(c), \quad B_1 = \overline{B}_1(c), \quad B_2 = \overline{B}_2(c),$$

while the integral constraints (3.7) are all satisfied. Observe that, depending on the values of the various parameters, the associated relative velocities $\overline{u}_i - c = \overline{\Psi}_{iz}$ may vanish at isolated values of z. These are "critical layers" where the flow reverses direction.

We write a general solution as a perturbation of the trivial solution, using lowercase letters for the perturbation variables:

$$\begin{aligned} \Psi_1 &=: \overline{\Psi}_1 + \psi_1, \quad M_1 =: \overline{M}_1 + m_1, \quad B_1 =: \overline{B}_1 + b_1, \\ \Psi_2 &=: \overline{\Psi}_2 + \psi_2, \quad M_2 =: \overline{M}_2 + m_2. \quad B_2 =: \overline{B}_2 + b_2. \end{aligned} \tag{3.10}$$

3.1.3. *Flattening transformations.* In the absence of the critical layers mentioned above, we could make a semi-Lagrangian change of variables originally due to Dubreil-Jacotin [14], using z as the dependent variable and Ψ_i as the independent variable. Indeed this transformation was used by Wang [41] for (a generalization of) our problem. Since we want to allow for critical layers, however, we are forced to use a less elegant change of coordinates, and we instead define the new vertical variable ζ by

$$\zeta := \begin{cases} -h_1 + \dfrac{h_1}{h_1+\eta_1}(h_1+z) & \text{if } -h_1 \le z \le \eta, \\ \dfrac{h_2}{h_2+\eta_2-\eta_1}(z-\eta_1) & \text{if } \eta_1 \le z \le h_2+\eta_2. \end{cases}$$

The change of variables $(x,y) \mapsto (x,\zeta)$ maps the lower and upper fluid layers D_1, D_2 onto the periodic strips

$$\Omega_1 = \mathbb{T}_\kappa \times (-h_1, 0), \quad \Omega_2 = \mathbb{T}_\kappa \times (0, h_2), \tag{3.11a}$$

where $\mathbb{T}_\kappa$ denotes the interval $[-\pi/\kappa, \pi/\kappa]$ with periodic boundary conditions. Similarly S_0, S_1, S_2 are sent to

$$\Gamma_0 = \mathbb{T}_\kappa \times \{\zeta = -h_1\}, \quad \Gamma_1 = \mathbb{T}_\kappa \times \{\zeta = 0\}, \quad \Gamma_2 = \mathbb{T}_\kappa \times \{\zeta = h_2\}. \tag{3.11b}$$

This change of variables is well-defined and piecewise smooth provided the inequalities

$$-h_1 < \eta_1 < h_2 + \eta_2 \tag{3.12}$$

hold so that the interface and free surface do not touch each other or the bed. Since we will be considering solutions where η_1, η_2 are small in $C^{2+\alpha}$, (3.12) will always hold.

For the remainder of the paper we will abuse notation and consider ψ_1, ψ_2 as functions of (x,ζ) rather than as functions of (x,z).

3.1.4. *Linearization.* Using the definitions in the previous two sections to change variables in (3.6)–(3.7) is tedious but straightforward, and we omit the calculations. Under the ever-present assumption (3.12), one obtains a system of equations for the unknown functions

$$\Phi = (\psi_1, \psi_2, \eta_1, \eta_2)$$

on the fixed domains Ω_1, Ω_2 and their boundaries $\Gamma_0, \Gamma_1, \Gamma_2$. The traveling-wave system (3.6) becomes

$$\begin{aligned}
\Delta\psi_1 &= N_1(\zeta, \Phi, D\Phi, D^2\Phi; c) && \text{in } \Omega_1, && (3.13a)\\
\Delta\psi_2 &= N_2(\zeta, \Phi, D\Phi, D^2\Phi; c) && \text{in } \Omega_2, && (3.13b)\\
\psi_1 - m_1 &= 0 && \text{on } \Gamma_0, && (3.13c)\\
\psi_1 - c_i\eta_1 &= N_4(\Phi; c) && \text{on } \Gamma_1, && (3.13d)\\
\psi_2 - c_i\eta_1 &= N_5(\Phi; c) && \text{on } \Gamma_1, && (3.13e)\\
\psi_2 - c_s\eta_2 - m_2 &= N_6(\Phi; c) && \text{on } \Gamma_2, && (3.13f)\\
-c_i\psi_{2\zeta} + \tilde{c}_i\psi_{1\zeta} + \beta_i\eta_1 - b_1 &= N_7(\Phi, D\Phi; c) && \text{on } \Gamma_1, && (3.13g)\\
-c_s\psi_{2\zeta} + \beta_s\eta_2 - b_2 &= N_8(\Phi, D\Phi; c) && \text{on } \Gamma_2, && (3.13h)
\end{aligned}$$

while the constraints (3.7) become

$$\int \eta_1 \, dx = 0, \tag{3.14a}$$

$$\int \eta_2 \, dx = 0, \tag{3.14b}$$

$$\int_{\Gamma_0} \psi_{1\zeta} \, dx = \int_{\Gamma_0} N_{11}(\Phi, D\Phi; c) \, dx, \tag{3.14c}$$

$$\int_{\Gamma_1} (\psi_{1\zeta} - \psi_{2\zeta}) \, dx = \int_{\Gamma_1} N_{12}(\Phi, D\Phi; c) \, dx. \tag{3.14d}$$

The functions N_i appearing on the right hand sides are each rational functions of their arguments and are well-defined and analytic in the region where (3.12) holds. They are genuinely nonlinear in that

$$\frac{\partial N_i}{\partial \Phi_j} = \frac{\partial N_i}{\partial (D_k \Phi_j)} = \frac{\partial N_i}{\partial (D_{k\ell} \Phi_j)} = 0 \qquad \text{whenever } (\Phi, D\Phi, D^2\Phi) = 0.$$

This much about the N_j can be deduced without writing them out explicitly; indeed the precise formulas will not be needed in this paper at all and so we omit them. The values of c-dependent coefficients on the left hand side of (3.13), on the other hand, are crucial:

$$\begin{aligned}
c_i &= c - \omega_1 h_1 = \text{relative speed at the interface},\\
c_s &= c - \omega_1 h_1 - \omega_2 h_2 = \text{relative wave speed at the surface},\\
\tilde{c}_i &= (1+r)c_i,\\
\beta_i &= c_s((1+r)\omega_1 - \omega_2) - gr = -gr + \overline{\Psi}_{2z}\overline{\Psi}_{2zz}(0) - (1+r)\overline{\Psi}_{1z}\overline{\Psi}_{1zz}(0),\\
\beta_s &= g - \omega_2 c_s = g + \overline{\Psi}_{2z}\overline{\Psi}_{2zz}(h_2).
\end{aligned} \tag{3.15}$$

Note that the coefficients β_s, β_i multiply the terms with the fewest derivatives in their respective equations, while $c_s, c_i, \tilde{c}_i$ multiply the highest order terms. Thus we expect qualitative properties such as Fredholm indices to be essentially independent of β_s, β_i. In the (at least formal) limit of a single homogeneous and irrotational layer, $c_s = c_i = \tilde{c}_i = c$ and $\beta_s = -\beta_i = g$.

3.1.5. *Abstract formulation and the Crandall–Rabinowitz theorem.* Fixing once and for all a Hölder parameter $\alpha \in (0,1)$, we work with the Banach spaces

$$\begin{aligned}
X &= C^{2+\alpha}_{\text{even}}(\Omega_1) \times C^{2+\alpha}_{\text{even}}(\Omega_2) \times C^{2+\alpha}_{\text{even}}(\Gamma_1) \times C^{2+\alpha}_{\text{even}}(\Gamma_2) \times \mathbb{R}^4,\\
Y &= V \times Z,\\
V &= C^{\alpha}_{\text{even}}(\Omega_1) \times C^{\alpha}_{\text{even}}(\Omega_2),\\
Z &= C^{2+\alpha}_{\text{even}}(\Gamma_0) \times [C^{2+\alpha}_{\text{even}}(\Gamma_1)]^2 \times C^{2+\alpha}_{\text{even}}(\Gamma_2) \times C^{1+\alpha}_{\text{even}}(\Gamma_1) \times C^{1+\alpha}_{\text{even}}(\Gamma_2) \times \mathbb{R}^4.
\end{aligned} \tag{3.16}$$

Here the subscript 'even' denotes evenness in the horizontal variable x; $2\pi/\kappa$-periodicity is already encoded in (3.11). We write elements of X as

$$U = (\Phi; \Lambda) = (\psi_1, \psi_2, \eta_1, \eta_2; b_1, b_2, m_1, m_2)$$

and elements of Y as

$$f = (f_1, f_2, \dots, f_{12}).$$

As mentioned in the previous subsection, the system (3.13) is only well-defined when the inequalities (3.12) hold. For this reason we will restrict our attention to the open subset

$$\mathscr{O} = \{U \in X : -h_1 < \eta_1 < h_2 + \eta_2\} \subset X,$$

which contains the axis $\{\Phi = 0\}$. We can then write (3.13)–(3.14) abstractly as

$$L(c)U = \mathscr{N}(U; c), \tag{3.17}$$

where

$$L(c) \colon X \to Y$$

is a bounded linear operator depending analytically on c and

$$\mathscr{N} : \mathscr{O} \times \mathbb{R} \to Y$$

is an analytic mapping between (open subsets of) Banach spaces. One can readily check that $L(c)$ and $\mathscr{N}$ preserve evenness and periodicity, at which point the above boundedness and analyticity are clear.

We will prove Theorem 3.1 by applying the following analytic version of the classical Crandall–Rabinowitz theorem [13].

Theorem 3.2 (Theorem 8.3.1 in [2]). *Let $\mathscr{L}(\lambda) \colon \mathscr{X} \to \mathscr{Y}$ be a bounded linear operator between Banach spaces depending analytically on a parameter $\lambda \in \mathbb{R}$, and let $\mathscr{N} : \mathscr{U} \to \mathscr{Y}$ be an analytic mapping defined on an open neighborhood $\mathscr{U}$ of $(0, \lambda_0)$ in $\mathscr{X} \times \mathbb{R}$ which is genuinely nonlinear in that $\mathscr{N}(0, \lambda) = 0$ and $\mathscr{N}_x(0, \lambda) = 0$ for all λ. If*

(i) *$\mathscr{L}(\lambda_0)$ is Fredholm with index zero;*
(ii) *$\ker \mathscr{L}(\lambda_0)$ is one-dimensional, spanned by some $\xi \in \mathscr{X}$; and*
(iii) *(transversality) $\mathscr{L}_\lambda(\lambda_0)\xi \notin \operatorname{ran} \mathscr{L}(\lambda_0)$,*

then $(0, \lambda_0)$ is a bifurcation point in the following sense. There exists $\varepsilon_0 > 0$ and a pair of analytic functions $(\tilde{x}, \tilde{\lambda}) \colon (-\varepsilon_0, \varepsilon_0) \to \mathscr{U}$ such that

(a) *$\mathscr{L}(\tilde{\lambda}(\varepsilon))\tilde{x}(\varepsilon) = \mathscr{N}(\tilde{x}(\varepsilon), \tilde{\lambda}(\varepsilon))$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$;*
(b) *$\tilde{x}(0) = 0$, $\tilde{\lambda}(0) = \lambda_0$, and $\tilde{x}'(0) = \xi$; and*
(c) *there exists an open neighborhood $\mathscr{V} \subset \mathscr{U}$ of $(0, \lambda_0)$ such that*

$$\big\{(x, \lambda) \in \mathscr{V} : \mathscr{L}(\lambda)x = \mathscr{N}(\lambda, x),\ x \neq 0\big\} = \big\{(\tilde{x}(\varepsilon), \tilde{\lambda}(\varepsilon)) : 0 < |\varepsilon| < \varepsilon_0\big\}.$$

3.2. **Fredholm index 0.** In this section we give sufficient conditions for the linear operator $L(c)$ in Section 3.1.5 to be Fredholm with index 0. Since we are treating an elliptic problem in a bounded domain, it is unsurprising that the index depends only on the inequalities

$$c_s \neq 0, \quad c_i \tilde{c}_i > 0 \tag{3.18}$$

and not on the lower-order coefficients β_s, β_i. For solitary wave problems the situation is far more delicate; see for instance [3]. It is useful to split $X = \tilde{X} \times \mathbb{R}^4$ and $Y = \tilde{Y} \times \mathbb{R}^4$ so that we can decompose L as the matrix operator

$$L =: \begin{pmatrix} T & S \\ R & 0 \end{pmatrix} : \tilde{X} \times \mathbb{R}^4 \to \tilde{Y} \times \mathbb{R}^4. \tag{3.19}$$

The genuinely infinite-dimensional part of the operator is then isolated in the upper-left entry T.

Lemma 3.3 (Invertibility)**.** *Suppose the inequalities* (3.18) *hold and moreover that* $\beta_s = \beta_i = 0$*. Then* $T\colon \tilde{X} \to \tilde{Y}$ *is invertible.*

Proof. Writing out the component equations of $T\Phi = f$, we have

$$\Delta\psi_1 = f_1 \quad \text{in } \Omega_1, \tag{3.20a}$$
$$\Delta\psi_2 = f_2 \quad \text{in } \Omega_2, \tag{3.20b}$$
$$\psi_1 = f_3 \quad \text{on } \Gamma_0, \tag{3.20c}$$
$$\psi_1 - c_i\eta_1 = f_4 \quad \text{on } \Gamma_1, \tag{3.20d}$$
$$\psi_2 - c_i\eta_1 = f_5 \quad \text{on } \Gamma_1, \tag{3.20e}$$
$$\psi_2 - c_s\eta_2 = f_6 \quad \text{on } \Gamma_2, \tag{3.20f}$$
$$-c_i\psi_{2\zeta} + \tilde{c}_i\psi_{1\zeta} = f_7 \quad \text{on } \Gamma_1, \tag{3.20g}$$
$$-c_s\psi_{2\zeta} = f_8 \quad \text{on } \Gamma_2. \tag{3.20h}$$

Subtracting (3.20e) and (3.20d), we obtain a transmission problem for (ψ_1, ψ_2) alone:

$$\begin{aligned} \Delta\psi_1 &= f_1 && \text{in } \Omega_1,\\ \Delta\psi_2 &= f_2 && \text{in } \Omega_2,\\ \psi_1 &= f_3 && \text{on } \Gamma_0,\\ \psi_2 - \psi_1 &= f_5 - f_4 && \text{on } \Gamma_1,\\ -c_i\psi_{2\zeta} + \tilde{c}_i\psi_{1\zeta} &= f_7 && \text{on } \Gamma_1,\\ -c_s\psi_{2\zeta} &= f_8 && \text{on } \Gamma_2. \end{aligned} \tag{3.21}$$

Thanks to the sign conditions (3.18), (3.21) can be solved uniquely for Ψ_1, Ψ_2, with the Schauder estimate [27]

$$\|\psi_1\|_{C^{2+\alpha}} + \|\psi_2\|_{C^{2+\alpha}} \le C\|f\|_Y, \tag{3.22}$$

where here and in what follows the constant C depends only on $c_s, c_i, \tilde{c}_i$ but can change from line to line. We can then uniquely solve (3.20d)–(3.20e) for η_1, η_2, with the obvious estimate

$$\|\eta_1\|_{C^{2+\alpha}} + \|\eta_2\|_{C^{2+\alpha}} \le C(\|\psi_1\|_{C^{2+\alpha}} + \|\psi_2\|_{C^{2+\alpha}} + \|f\|_Y). \tag{3.23}$$

Combining (3.22) and (3.23) leads at once to the Schauder estimate $\|\Phi\|_Y \le C\|f\|_Y$. □

Corollary 3.4 (Fredholm index 0)**.** *If the inequalities* (3.18) *hold then* $T\colon X \to Y$ *and* $L\colon X \to Y$ *are Fredholm with index 0.*

Proof. Writing the dependence on β_s, β_i explicitly, we can decompose T as

$$T = T_0 + \beta_s T_1 + \beta_i T_2.$$

The first term T_0 is invertible by Lemma 3.3. Since T_1, T_2 are compact, we deduce that T is Fredholm with index 0. Since the factors of $\mathbb{R}^4$ have the same dimension in $X = \tilde{X} \times \mathbb{R}^4$ and $Y = \tilde{Y} \times \mathbb{R}$, the full operator L is then also Fredholm with index zero by the Fredholm bordering lemma [35]. □

3.3. **An abstract lemma.** While the Fredholm index of $L(c)$ only depends on the structural inequalities (3.18), the remaining hypotheses in the Crandall–Rabinowitz theorem 3.2 require more detailed information. If $L(c)$ were a Fourier multiplier acting on a single function of a single variable, the way forward would be clear, and indeed [30, 26] are able reformulate their nonlinear problems so that this is the case. Rather than pursue similar reductions here (but now to vector-valued functions of a single variable), we treat the original operator $L(c)$ directly, using the abstract lemma below as our primary tool.

The general setting is the following. We have a family of operators $L(t)\colon X \to Y$ which we cannot easily express in terms of operators on finite-dimensional spaces (i.e., we can Fourier transform in x, but we are still left with inhomogeneous ODEs in ζ). This problem disappears, however, if we suitably restrict the domain and range of $L(t)$ by considering a composition $\Pi_V L(t)E(t)\colon W \to Z$ (i.e., if we set the inhomogeneous terms in the ODEs to zero and express everything in terms of boundary data). The question is then what we can conclude about the full operators $L(t)$ by studying the simpler operators $\Pi_V L(t)E(t)$.

More precisely, suppose we have smooth families of bounded linear operators $L(t)$ and $E(t)$ between Banach spaces that fit into the following diagram:

$$W \xrightarrow{E(t)} X \xrightarrow{L(t)} Y = V \times Z.$$

Letting Π_Z, Π_V be the projections of Y onto its factors, we require

$$\operatorname{ran} E = \ker \Pi_V L, \qquad \ker E = \{0\}. \tag{3.24}$$

Moreover we suppose that for each $\ell \in \mathbb{N}$ there are t-independent projections P_ℓ, Q_ℓ and isomorphisms I_ℓ, J_ℓ such that

$$W \xrightarrow{P_\ell} P_\ell W \xrightarrow{I_\ell} \mathbb{R}^{n_\ell}, \qquad Z \xrightarrow{Q_\ell} Q_\ell Z \xrightarrow{J_\ell} \mathbb{R}^{n_\ell}$$

for some finite dimension n_ℓ depending only on ℓ, and that these projections diagonalize $\Pi_Z LE$ in that

$$\sum_{\ell=0}^{\infty} P_\ell w = w, \qquad \sum_{\ell=0}^{\infty} Q_\ell z = z \tag{3.25a}$$

for each fixed $w \in W$ and $z \in Z$, and

$$Q_j \Pi_Z LEP_\ell = 0,\ Q_j Q_\ell = 0,\ P_j P_\ell = 0 \quad \text{for } j \neq \ell. \tag{3.25b}$$

The following result says that certain properties of $L(t)$ can sometimes be inferred from related properties of the $n_\ell \times n_\ell$ matrices

$$M_\ell(t) = J_\ell Q_\ell \Pi_Z L(t) E(t) I_\ell^{-1}. \tag{3.26}$$

Lemma 3.5. *Suppose that for some ℓ_* and t_* the following hold:*

(i) $\ker M_{\ell_*}(t_*) = \operatorname{span}\{\mu\}$ *is one-dimensional;*
(ii) $M_\ell(t_*)$ *is invertible for $\ell \neq \ell_*$; and*
(iii) $\left.\dfrac{d}{dt}\right|_{t=t_*} \det M_{\ell_*}(t) \neq 0.$

Then

(a) $\ker L(t_*) = \operatorname{span}\{\xi\}$ *where $\xi = EI_{\ell^*}^{-1}\mu$; and*
(b) $L'(t_*)\xi \notin \operatorname{ran} L(t_*)$.

Note that we are neither assuming nor proving that $L(t_*)$ is Fredholm with index 0. Also, while (i)–(ii) are more or less equivalent to (a), we do not in general expect (b) to imply (iii).

Condition (iii) in Lemma 3.5 comes from the following finite-dimensional lemma.

Lemma 3.6 (Transverality in finite dimensions)**.** *Let M, M' be complex $n \times n$ matrices and assume that* $\ker M = \operatorname{span}\{\mu\}$ *is one-dimensional. Then* $M'\mu \in \operatorname{ran} M$ *if and only if*

$$\frac{d}{dt}\bigg|_{t=0} \det(M + tM') = 0. \tag{3.27}$$

Proof. Without loss of generality we can assume that M is in Jordan normal form, i.e. that

$$M = \begin{pmatrix} A & 0 \\ 0 & J \end{pmatrix},$$

where A is an invertible $\ell \times \ell$ matrix and J is a $(n-\ell)\times(n-\ell)$ Jordan block with 0's down the diagonal. Then $\ker M$ is spanned by $\mu = e_{\ell+1}$ while $\operatorname{ran} M = \operatorname{span}\{e_n\}^\perp$, and so $M'\mu \in \operatorname{ran} M$ if and only if

$$e_n \cdot (M' e_{\ell+1}) = M'_{n,\ell+1} = 0. \tag{3.28}$$

Expanding the determinant we find

$$\det(M + tM') = \det\left(\begin{pmatrix} A & 0 \\ 0 & J \end{pmatrix} + tM'\right) = t\det(A) M'_{\ell+1,n} + O(t^2).$$

Comparing with (3.28) we see that $M'\mu \in \operatorname{ran} M$ is equivalent to (3.27) as desired. □

Proof of Lemma 3.5. First we show (a). Since $t = t_*$ throughout, we suppress dependence on it. The assumption $\Pi_V LE = 0$ gives at once that $\Pi_V L\xi = 0$, and hence $L\xi = 0$ follows from the calculation

$$\Pi_Z L\xi = \sum_j Q_j \Pi_Z LEP_{\ell_*} I_{\ell_*}^{-1}\mu = Q_{\ell_*}\Pi_Z LEP_{\ell_*} I_{\ell_*}^{-1}\mu = J_{\ell_*}^{-1} M_{\ell^*}\mu = 0$$

in which we have used (3.25) and (i). Conversely, suppose that $x \in \ker L$. Then (3.24) implies $x = Ew$ for some $w \in W$. By (3.25a) we can then write

$$x = Ew = \sum_\ell EP_\ell w,$$

so that applying (3.25a) again and using (3.25b) yields

$$0 = \Pi_V LEw = \sum_\ell \sum_m Q_\ell \Pi_V LEP_m w = \sum_\ell Q_\ell(\Pi_V LEP_\ell w).$$

By (3.25b) each term in this sum must vanish,

$$M_\ell(I_\ell P_\ell w) = 0 \text{ for all } \ell.$$

Our assumption (ii) therefore implies $P_\ell w = 0$ for $\ell \neq \ell_*$, while (i) gives $I_{\ell_*} P_{\ell_*} w \in \operatorname{span}\{\mu\}$. This in turn implies $x = EI_{\ell^*}^{-1} w \in \operatorname{span}\{\xi\}$ as desired.

It remains to show (b). Again L, E, L', E' will always be evaluated at $t = t_*$, and so we suppress this dependence for readability. Suppose that $x \in X$ solves $Lx = L'\xi$. We must show that (iii) does not hold. Setting $\omega = I_{\ell_*}^{-1}\mu$, we calculate

$$L(x + E'\omega) = L'E\omega + LE'\omega = (LE)'\omega. \tag{3.29}$$

Differentiating the assumption $\Pi_V LE = 0$, we find that $\Pi_V(LE)' = 0$. Applying Π_V to (3.29) therefore yields $x + E'\omega \in \ker \Pi_V L = \operatorname{ran} E$. Thus we can write

$$x + E'\omega = Ew$$

for some $w \in W$. We now apply $J_{\ell_*} Q_{\ell_*} \Pi_Z$ to both sides of (3.29) and compare the results. On the left hand side (3.25) implies

$$\begin{aligned} J_{\ell_*} Q_{\ell_*} \Pi_Z L(x + E'\omega) &= J_{\ell_*} Q_{\ell_*} \Pi_Z LEw \\ &= J_{\ell_*} Q_{\ell_*} \Pi_Z LEP_{\ell_*} w \\ &= M_{\ell_*}(I_{\ell_*} P_{\ell_*} w) \in \operatorname{ran} M_{\ell_*}, \end{aligned} \tag{3.30}$$

while on the right hand side we get

$$J_{\ell_*} Q_{\ell_*} \Pi_Z (LE)'\omega = J_{\ell_*} Q_{\ell_*} \Pi_Z (LE)' I_{\ell_*}^{-1} \mu = M'_{\ell_*} \mu. \tag{3.31}$$

Combining (3.29)–(3.31) yields $M'_{\ell_*}\mu \in \operatorname{ran} M_{\ell_*}$. Applying Lemma 3.6 with $M = M_{\ell_*}$ and $M' = M'_{\ell_*}$, we conclude that (iii) does not hold, and the proof is complete. □

3.4. **Application of the lemma.** We now apply Lemma 3.5 to the linear operator $L(c)\colon X \to Y$ appearing in our problem. We decompose $Y = V \times Z$ exactly as in (3.16), and set

$$W = \big(C^{2+\alpha}_{\text{even}}(\Gamma_0) \times [C^{2+\alpha}(\Gamma_1)]^2 \times C^{2+\alpha}(\Gamma_2)\big) \times \big(C^{2+\alpha}(\Gamma_0) \times C^{2+\alpha}(\Gamma_1)\big) \times \mathbb{R}^4,$$

where the first four factors will represent the boundary values of the functions ψ_1, ψ_2 ordered from bottom to top, i.e. $t_1 = \psi_1|_{\Gamma_0}$, $t_2 = \psi_1|_{\Gamma_1}$, $t_3 = \psi_2|_{\Gamma_1}$, $t_4 = \psi_2|_{\Gamma_2}$. Writing elements of W as

$$w = (t_1, t_2, t_3, t_4, \eta_1, \eta_2, b_1, b_2, m_1, m_2),$$

our mapping $E\colon W \to X$ is independent of c and defined by

$$Ew = (\psi_1, \psi_2, \eta_1, \eta_2, b_1, b_2, m_1, m_2)$$

where ψ_1, ψ_2 are the unique solutions of the Dirichlet problems

$$\begin{cases} \Delta\psi_1 = 0 & \text{in } \Omega_1, \\ \psi_1 = t_1 & \text{on } \Gamma_0, \\ \psi_1 = t_2 & \text{on } \Gamma_1, \end{cases} \qquad \begin{cases} \Delta\psi_2 = 0 & \text{in } \Omega_2, \\ \psi_2 = t_3 & \text{on } \Gamma_1, \\ \psi_2 = t_4 & \text{on } \Gamma_2. \end{cases}$$

The boundedness and injectivity of E follows from standard elliptic theory. Moreover $\ker \Pi_V L = \operatorname{ran} E$ by construction and so (3.24) holds.

The projections P_ℓ, Q_ℓ and isomorphisms I_ℓ, J_ℓ are defined in terms of Fourier coefficients, where $\ell \in \mathbb{N}$ corresponds to a wavenumber $k = \ell\kappa$. Adopting the convention

$$\mathcal{F}_\ell f := \begin{cases} \dfrac{\kappa}{\pi} \displaystyle\int_{-\pi/\kappa}^{\pi/\kappa} f(x)\cos(\ell\kappa x)\,dx & \ell = 1, 2, 3, \ldots, \\ \dfrac{\kappa}{2\pi} \displaystyle\int_{-\pi/\kappa}^{\pi/\kappa} f(x)\,dx & \ell = 0, \end{cases} \tag{3.32}$$

we abuse notation slightly and set

$$P_\ell = \cos(\ell\kappa x)\mathcal{F}_\ell, \qquad Q_\ell = \cos(\ell\kappa x)\mathcal{F}_\ell. \tag{3.33}$$

The hypotheses in (3.25) now follow by familiar properties of Fourier series.

When $\ell \neq 0$, the last four components of $P_\ell w$ and $Q_\ell f$ vanish because they are nonzero Fourier modes of constant functions. Thus the relevant dimension is $n_\ell = 6$ and the isomorphisms $I_\ell \colon P_\ell W \to \mathbb{R}^6$ and $J_\ell \colon P_\ell Z \to \mathbb{R}^6$ drop the last four components of their arguments:

$$\begin{aligned} I_\ell(t_1, t_2, t_3, t_4, \eta_1, \eta_2, b_1, b_2, m_1, m_2) &= \mathcal{F}_\ell(t_1, t_2, t_3, t_4, \eta_1, \eta_2), \\ J_\ell(f_3, f_4, \dots, f_{12}) &= \mathcal{F}_\ell(f_3, f_4, \dots, f_8). \end{aligned}$$

When $\ell = 0$, the relevant dimension is $n_0 = 10$ and the isomorphisms are simply $I_0 = \mathcal{F}_0$ and $J_0 = \mathcal{F}_0$.

All that is left to do to apply Lemma 3.5 is to calculate the matrices

$$M_\ell(c) = J_\ell Q_\ell \Pi_Z L(c) E I_\ell^{-1} \tag{3.34}$$

and to study their kernels and determinants. Fix $\ell \neq 0$, set $k = \ell\kappa$, and consider a generic element

$$w_\ell = (\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{t}_4, \hat{\eta}_1, \hat{\eta}_2) \in \mathbb{R}^6.$$

Then $w = I_\ell^{-1} w_\ell$ is given by

$$w = I_\ell^{-1} w_\ell = (\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{t}_4, \hat{\eta}_1, \hat{\eta}_2, 0, 0, 0, 0)\cos(kx) \in P_\ell W,$$

and we easily check that

$$Ew = (\hat{\psi}_1, \hat{\psi}_2, \hat{\eta}_1, \hat{\eta}_2, 0, 0, 0, 0)\cos(kx) \in X,$$

where

$$\begin{aligned} \hat{\psi}_1 &= \frac{\sinh k(\zeta + h_1)}{\sinh kh_1}\hat{t}_2 - \frac{\sinh k\zeta}{\sinh kh_1}\hat{t}_1, \\ \hat{\psi}_2 &= \frac{\sinh k\zeta}{\sinh kh_2}\hat{t}_4 - \frac{\sinh k(\zeta - h_2)}{\sinh kh_2}\hat{t}_3. \end{aligned} \tag{3.35}$$

In particular,

$$\begin{aligned} \hat{\psi}_{1\zeta}|_{\zeta=0} &= \hat{t}_2 k \coth kh_1 - \hat{t}_1 k \operatorname{csch} kh_1, \\ \hat{\psi}_{2\zeta}|_{\zeta=0} &= \hat{t}_4 k \operatorname{csch} kh_2 - \hat{t}_3 k \coth kh_2, \\ \hat{\psi}_{2\zeta}|_{\zeta=h_2} &= \hat{t}_4 k \coth kh_2 - \hat{t}_3 k \operatorname{csch} kh_2. \end{aligned} \tag{3.36}$$

Applying the operator L (see the left hand side of (3.13)) and collecting terms, we find that the matrix M_ℓ defined in (3.34) is

$$M_\ell = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -c_i & 0 \\ 0 & 0 & 1 & 0 & -c_i & 0 \\ 0 & 0 & 0 & 1 & 0 & -c_s \\ -\tilde{c}_i k \operatorname{csch} kh_1 & \tilde{c}_i k \coth kh_1 & c_i k \coth kh_2 & -c_i k \operatorname{csch} kh_2 & \beta_i & 0 \\ 0 & 0 & c_s k \operatorname{csch} kh_2 & -c_s k \coth kh_2 & 0 & \beta_s \end{pmatrix}.$$

For $\ell \neq 0$, we instead take a generic element $w_0 \in \mathbb{R}^{10}$ of the form

$$w_0 = (\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{t}_4, \hat{\eta}_1, \hat{\eta}_2, b_1, b_2, m_1, m_2) \in \mathbb{R}^{10}$$

and find that

$$EI_0^{-1} w_0 = (\hat{\psi}_1, \hat{\psi}_2, \hat{\eta}_1, \hat{\eta}_2, b_1, b_2, m_1, m_2) \in X,$$

where

$$\hat{\psi}_1 = \frac{\zeta + h_1}{h_1}\hat{t}_2 - \frac{\zeta}{h_1}\hat{t}_1, \qquad \hat{\psi}_2 = \frac{\zeta}{h_2}\hat{t}_4 - \frac{\zeta - h_2}{h_2}\hat{t}_3. \tag{3.37}$$

Applying L as before we obtain the 10×10 matrix

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -c_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -c_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -c_s & 0 & 0 & 0 & -1 \\ -\tilde{c}_i/h_1 & \tilde{c}_i/h_1 & c_i/h_2 & -c_i/h_2 & \beta_i & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & c_s/h_2 & -c_s/h_2 & 0 & \beta_s & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1/h_1 & 1/h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/h_1 & 1/h_1 & 1/h_2 & -1/h_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Lemma 3.7. *Suppose that (3.18) holds. Then the matrix M_0 is invertible, while $\det M_\ell = -d(\ell\kappa, c)$ so that M_ℓ is invertible if and only if $d(\ell\kappa, c) \neq 0$. Moreover, the kernel of M_ℓ is at most one-dimensional.*

Proof. An explicit calculation shows that (even without (3.18))

$$\det M_0 = \frac{1}{h_1 h_2} \neq 0.$$

Now fix $\ell \neq 0$ and set $k = \ell\kappa$. Since the upper 4×4 block of M_ℓ is the identity, the usual arguments for block matrices show that its kernel has the same dimension as the 2×2 matrix

$$\tilde{M}_\ell = \begin{pmatrix} c_i\tilde{c}_i k \coth kh_1 + c_i^2 k \coth kh_2 + \beta_i & c_s c_i k \operatorname{csch} kh_2 \\ c_i c_s k \operatorname{csch} kh_2 & c_s^2 k \coth kh_2 - \beta_s \end{pmatrix} \tag{3.38}$$

obtained by subtracting the product of its bottom-left 2×4 block and its upper-right 4×2 block from its bottom-right 2×2 block and then flipping the sign of the first column. Similarly

$$\det M_\ell = -\det \tilde{M}_\ell = -d(\ell\kappa, c)$$

where $d(k, c)$ was defined in (3.1). Thanks to (3.18) and $k > 0$, the upper-right entry of $\tilde{M}_\ell$ is nonzero, and so its kernel is at most one-dimensional. □

We are now finally in a position to prove our main existence result.

Proof of Theorem 3.1. Suppose that c_*, κ satisfy hypotheses (i)–(iii) of the theorem. By (iii), the corresponding values of $c_s, c_i, \tilde{c}_i$ satisfy (3.18), and so $L(c_*)$ is Fredholm with index 0 by Lemma 3.4. Applying Lemma 3.7 we get that M_ℓ is invertible for $\ell \neq 1$ while M_1 has a one-dimensional kernel. Moreover by hypothesis (i) of the theorem we have

$$\frac{d}{dc}\bigg|_{c=c_*} \det M_1(c) = -\frac{\partial d}{\partial c}(\kappa, c_*) \neq 0.$$

Thus all of the hypotheses of Lemma 3.5 are satisfied, and hence $\ker L(c_*) = \operatorname{span}\{\xi\}$ is one-dimensional and the transversality condition $L_c(c_*)\xi \notin \operatorname{ran} L(c_*)$ holds. This in turn means that the hypotheses of Theorem 3.2 are satisfied, and hence that we have a unique curve of solutions to our nonlinear problem (3.17). The constraints

(3.3)–(3.5) are built into our formulation of the problem, and are hence satisfied automatically.

It remains to justify the expansions (3.2). Let $\xi \in \ker L(c_*)$. By Lemma 3.5 we have $\xi = EI_1^{-1}\mu$ where $\mu = (\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{t}_4, \hat{\eta}_1, \hat{\eta}_2) \in \ker M_1$. Block matrix calculations with M_1 similar to those in the proof of Lemma 3.7 show that this implies $(\hat{\eta}_1, \hat{\eta}_2) \in \ker \tilde{M}_1$. We claim that the entry $(\tilde{M}_1)_{22}$ of this matrix is nonzero. Indeed, if it were zero then we would have $\det \tilde{M}_1 = (c_s c_i k \operatorname{csch} kh_2)^2 \neq 0$. Thus we can assume without loss of generality that our element of the kernel has $\hat{\eta}_1 = 1$ and

$$\hat{\eta}_2 = -\frac{(\tilde{M}_1)_{12}}{(\tilde{M}_1)_{22}} = -\frac{c_s c_i \kappa \operatorname{csch} \kappa h_2}{c_s^2 \kappa \coth \kappa h_2 + c_s \omega_2 - g}.$$

Thus $\xi = EI_1^{-1}\mu = (\dot{\psi}_1, \dot{\psi}_2, \dot{\eta}_1, \dot{\eta}_2, \dot{b}_1, \dot{b}_2, \dot{m}_1, \dot{m}_2)$ where

$$\dot{\eta}_1 = \cos(\kappa x), \qquad \dot{\eta}_2 = -\frac{c_s c_i \kappa \operatorname{csch} \kappa h_2}{c_s^2 \kappa \coth \kappa h_2 + c_s \omega_2 - g} \cos(\kappa x).$$

The first two lines of (3.2) are then simply Theorem 3.2(b). The fact that $c - c_* = O(\varepsilon^2)$ follows from the fact that our nonlinear problem (3.17) is preserved by the transformation $x \mapsto x + \pi/\kappa$; see for instance remark 4.8 in [12]. □

4. **Generalizations and other parametrizations.** In this final section we discuss how the methods of Section 3 can be applied to a variety of related bifurcation problems.

4.1. **Coriolis forces.** Thanks to Proposition 2.1, our existence result Theorem 3.1 immediately implies an existence result for waves with nonzero Coriolis parameter Ω. On the other hand, the waves along the resulting bifurcation curve will have different values of the gravitational constant g, which may not be desirable in applications.

Nevertheless, we can modify our proof of Theorem 3.1 so that $\Omega \neq 0$ and g are both held constant. By Proposition 2.1, we can accommodate $\Omega \neq 0$ simply by replacing g by $g - 2\Omega c$ in (3.13)–(3.14). This changes the nonlinear terms in unimportant ways, and affects the linear terms only through the lower-order coefficients β_s, β_i. Thus the Fredholm index arguments in Section 3.2 and the calculations in Section 3.4 are unaffected, except of course that g must be replaced by $g - 2\Omega c$ in the dispersion relation $d(k, c) = 0$. Defining

$$\begin{aligned} d^\Omega(k,c) = \Big[&\Big(c_i^2 k\big((1+r)\coth kh_1 + \coth kh_2\big) + c_i((1+r)\omega_1 - \omega_2) - (g - 2\Omega c) r \Big) \\ &\times \Big(c_s^2 k \coth kh_2 + c_s \omega_2 - (g - 2\Omega c) \Big) \Big] - \Big(c_s c_i k \operatorname{csch} kh_2 \Big)^2, \end{aligned}$$

we therefore have the following corollary.

Corollary 4.1. *Fix $\kappa, h_1, h_2, r, \omega_1, \omega_2, g, \Omega$. Suppose that at some speed c_* we have*

(i) (Simple root) $d^\Omega(\kappa, c_*) = 0$ *and* $d_c^\Omega(\kappa, c_*) \neq 0$*;*
(ii) (Non-resonance) $d^\Omega(\ell\kappa, c_*) \neq 0$ *for* $\ell \neq \pm 1, 0$*; and*
(iii) (Non-critical surface and interface) $c_* \neq \omega_1 h_2, \omega_1 h_1 + \omega_2 h_2$.

Then there is an analytic curve of solutions to (1.4), parametrized by a small parameter ε, and satisfying (3.2)–(3.5) except that the asymptotic expansion for η_2 is replaced by

$$\eta_2 = -\varepsilon \frac{c_s c_i \kappa \operatorname{csch} \kappa h_2}{c_s^2 \kappa \coth \kappa h_2 + c_s \omega_2 - (g - 2\Omega c)} \cos(\kappa x) + O(\varepsilon^2).$$

4.2. **Wave number as the bifurcation parameter.** We have chosen to keep the basic wave number κ constant and used c as a bifurcation parameter, but these roles can be reversed. To avoid having parameter-dependent domains, we switch to a scaled horizontal variable $\tilde{x} = x/\kappa$. This replaces the tori $\mathbb{T}_\kappa$ in (3.11) with $\mathbb{T}_1$ at the cost of replacing the Laplacian Δ in (3.13) (and hence in $L(\kappa)$) with the κ-dependent operator $\kappa^2\partial_{\tilde{x}}^2 + \partial_\zeta^2$. Of course the nonlinear terms N_j are modified as well. Defining X, Y, Z, V, W as before, the extension operator $E(\kappa)\colon W \to X$ is now defined in terms of the Dirichlet problems

$$\begin{cases} (\kappa^2\partial_{\tilde{x}}^2 + \partial_\zeta^2)\psi_1 = 0 & \text{in } \Omega_1, \\ \psi_1 = t_1 & \text{on } \Gamma_0, \\ \psi_1 = t_2 & \text{on } \Gamma_1, \end{cases} \qquad \begin{cases} (\kappa^2\partial_{\tilde{x}}^2 + \partial_\zeta^2)\psi_2 = 0 & \text{in } \Omega_2, \\ \psi_2 = t_3 & \text{on } \Gamma_1, \\ \psi_2 = t_4 & \text{on } \Gamma_2, \end{cases}$$

and we replace κ by 1 in the definitions (3.32) and (3.33) of the projections P_ℓ, Q_ℓ. Keeping the shorthand $k = \ell\kappa$, the matrices M_ℓ and M_0 are unaffected, except that they are now viewed as functions of $\kappa = k/\ell$ rather than c. This leads to the following analogue of Theorem 3.1.

Corollary 4.2. *Define $d(k, c)$ as in (3.1), and fix $c, h_1, h_2, r, \omega_1, \omega_2, g$. Suppose that at some wave number κ_* we have*

(i) (Simple root) $d(\kappa_*, c) = 0$ *and* $d_\kappa(\kappa_*, c) \neq 0$*;*
(ii) (Non-resonance) $d(\ell\kappa_*, c) \neq 0$ *for* $\ell \neq \pm 1, 0$*; and*
(iii) (Non-critical surface and interface) $c_* \neq \omega_1 h_2, \omega_1 h_1 + \omega_2 h_2$.

Then there is an analytic curve of solutions to (1.4) satisfying (3.3)–(3.5), with the asymptotic expansions

$$\begin{aligned} \eta_1(x/\kappa) &= \varepsilon\cos(x) + O(\varepsilon^2), \\ \eta_2(x/\kappa) &= \varepsilon - \frac{c_s c_i \kappa \operatorname{csch} \kappa h_2}{c_s^2 \kappa \coth \kappa h_2 + c_s\omega_2 - g}\cos(x) + O(\varepsilon^2), \\ \kappa &= \kappa_* + O(\varepsilon^2). \end{aligned}$$

4.3. **Non-constant vorticity.** In Theorem 3.1 our solutions are perturbations of the "trivial" stream functions (3.9) representing a piecewise-linear shear flow. Much more general shear flows can also in principle be treated. To avoid getting lost in technical issues outside the scope of the present paper, we only sketch the ideas and do not state any precise results.

For simplicity consider the case where the speed c is fixed and κ is the bifurcation parameter as above. In place of (3.9) suppose that we are given trivial stream functions $\overline{\Psi}_1(z)$ and $\overline{\Psi}_2(z)$ satisfying

$$\overline{\Psi}_1(0) = \overline{\Psi}_2(0) = 0, \qquad \overline{\Psi}_{1z}(0) = \overline{\Psi}_{2z}(0), \qquad \overline{\Psi}_{1z}(-h_1) = 0,$$

as well as ordinary differential equations

$$\overline{\Psi}_{1zz} = \gamma_1(\overline{\Psi}_1), \qquad \overline{\Psi}_{2zz} = \gamma_2(\overline{\Psi}_2) \tag{4.1}$$

for some smooth vorticity functions $\gamma_1, \gamma_2\colon \mathbb{R} \to \mathbb{R}$. To avoid technicalities with the ansatz (3.10), we assume $\overline{\Psi}_1$ is defined and solves (4.1) on an open neighborhood of $[-h_1, 0]$ and similarly for $\overline{\Psi}_2$. The first two lines of (3.6) now become

$$\begin{aligned} \Delta\Psi_1 &= \gamma_1(\Psi_1) && \text{in } D_1, \\ \Delta\Psi_2 &= \gamma_2(\Psi_2) && \text{in } D_2, \end{aligned}$$

and hence the first two lines of (3.13) become

$$\begin{aligned}(\kappa^2\partial_{\tilde x}^2+\partial_\zeta^2-\gamma_1'(\Psi_1(\zeta)))\psi_1 &= N_1(\zeta,\Phi,D\Phi,D^2\Phi;c) \quad \text{in } \Omega_1,\\ (\kappa^2\partial_{\tilde x}^2+\partial_\zeta^2-\gamma_2'(\Psi_1(\zeta)))\psi_2 &= N_2(\zeta,\Phi,D\Phi,D^2\Phi;c) \quad \text{in } \Omega_2.\end{aligned} \tag{4.2}$$

The remaining lines in (3.13) and (3.14) are the same, except that the formulas (3.15) for the coefficients are now

$$\begin{aligned} c_i &= -\overline{\Psi}_{1z}(0) = -\overline{\Psi}_{2z}(0),\\ c_s &= -\overline{\Psi}_{2z}(h_1),\\ \tilde c_i &= (1+r)c_i,\\ \beta_i &= -gr + \overline{\Psi}_{2z}\overline{\Psi}_{2zz}(0) - (1+r)\overline{\Psi}_{1z}\overline{\Psi}_{1zz}(0),\\ \beta_s &= g + \overline{\Psi}_{2z}\overline{\Psi}_{2zz}(h_2).\end{aligned}$$

The operator $E(\kappa)$ is defined in terms of the Dirichlet problems

$$\begin{cases}(\kappa^2\partial_{\tilde x}^2+\partial_\zeta^2-\gamma_1'(\Psi_1))\psi_1 = 0 \ \text{ in } \Omega_1,\\ \psi_1 = t_1 \text{ on } \Gamma_0,\\ \psi_1 = t_2 \text{ on } \Gamma_1,\end{cases} \qquad \begin{cases}(\kappa^2\partial_{\tilde x}^2+\partial_\zeta^2-\gamma_2'(\Psi_1))\psi_2 = 0 \ \text{ in } \Omega_2,\\ \psi_2 = t_3 \text{ on } \Gamma_1,\\ \psi_2 = t_4 \text{ on } \Gamma_2\end{cases}$$

which have unique solutions for κ outside a (possibly empty) discrete set. This gives considerably less explicit analogues of (3.35) and (3.36), leading to similarly implicit formulas for the matrices M_ℓ, their determinants, and ultimately to a dispersion relation $d^{\overline{\Psi}_1,\overline{\Psi}_2}(\kappa,c)=0$.

4.4. **The Boussinesq limit.** As mentioned in the introduction, the free-surface boundary condition treated in Theorem 3.1 is more complicated than the rigid-lid condition used in [40, 30], as can be appreciated by inspecting the rather complicated dispersion relation (3.1). When studying internal waves with $|\eta_2| \ll |\eta_1|$, the rigid-lid problem is often put forward as a reasonable approximation of the free-surface problem.

One systematic way to derive a rigid-lid-type model from the free-surface problem is to make a Boussinesq approximation. Here the dimensionless density ratio $r = (\rho_1-\rho_2)/\rho_2 > 0$ is used as a small parameter, while the reduced gravity $g' = gr$ is held constant. Sending $r \to 0$ does not affect (3.6a)–(3.6f), but the dynamic boundary conditions (3.6g)–(3.6h) become

$$\begin{aligned}\tfrac12|\nabla\Psi_2|^2 - \tfrac12|\nabla\Psi_1|^2 + g'\eta_1 &= B_1 \text{ on } S_1,\\ \eta_2 &= 0 \quad \text{on } S_2,\end{aligned}$$

so that in particular the free surface S_N is flat. One can analyze the resulting nonlinear problem for (Ψ_1,Ψ_2,η_1) using the techniques of this paper; indeed the calculations are considerably simpler. However the number and nature of the boundary conditions has changed, as well as the number of unknowns, and so the spaces X, Y, etc., must all be changed. As can be guessed by sending $r \to 0$ in (3.1) with $g' = gr$ fixed, the dispersion relation is $d^{\mathrm{Bous}}(k,c) = 0$ where

$$d^{\mathrm{Bous}}(k,c) = c_i^2 k\big(\coth kh_1 + \coth kh_2\big) + c_i(\omega_1-\omega_2) - g'. \tag{4.4}$$

Unlike (3.1), this is a quadratic function of c, and more importantly it is a strictly increasing function of $k > 0$. Thus the existence result can dispense with several of the hypotheses in Theorem 3.1:

Corollary 4.3. *Define* $d^{\text{Bous}}(k,c)$ *as above, and fix* $c, h_1, h_2, \omega_1, \omega_2, g'$. *Suppose that at some wave number* $\kappa_* \neq 0$ *we have*

(i) (Root) $d^{\text{Bous}}(\kappa_*, c) = 0$*; and*
(ii) (Non-critical interface) $c_* \neq \omega_1 h_2$.

Then there is an analytic curve of solutions of the above Boussinesq system, satisfying (3.2)–(3.5) *except that* $\eta_2 \equiv 0$.

For a non-rigorous study of the above Boussinesq approximation in the context of the Equatorial Undercurrent, see [42]. An interesting mathematical question is to what extent this limit can be made rigorous. For instance, can the solutions in Theorem 3.1 be constructed uniformly in a neighborhood of $r = 0$ with a fixed g'? Since this is a singular limit (the dynamic boundary condition (3.6h) changes type), any uniform construction will likely involve the introduction of boundary layers supported near the free surface.

REFERENCES

[1] Alexios Aivaliotis. On the symmetry of equatorial travelling water waves with constant vorticity and stagnation points. *Nonlinear Anal. Real World Appl.*, 34:159–171, 2017.

[2] Boris Buffoni and John Toland. *Analytic theory of global bifurcation*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2003. An introduction.

[3] Robin Ming Chen, Samuel Walsh, and Miles H. Wheeler. Existence and qualitative theory for stratified solitary water waves. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 35(2):517–576, 2018.

[4] Alan Compelli and Rossen Ivanov. On the dynamics of internal waves interacting with the equatorial undercurrent. *J. Nonlinear Math. Phys.*, 22(4):531–539, 2015.

[5] Alan Compelli and Rossen Ivanov. The dynamics of flat surface internal geophysical waves with currents. *J. Math. Fluid Mech.*, 19(2):329–344, 2017.

[6] A. Constantin, R. I. Ivanov, and C.-I. Martin. Hamiltonian formulation for wave-current interactions in stratified rotational flows. *Arch. Ration. Mech. Anal.*, 221(3):1417–1447, 2016.

[7] A. Constantin and R. S. Johnson. The dynamics of waves interacting with the Equatorial Undercurrent. *Geophys. Astrophys. Fluid Dyn.*, 109(4):311–358, 2015.

[8] Adrian Constantin. *Nonlinear water waves with applications to wave-current interactions and tsunamis*, volume 81 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

[9] Adrian Constantin. On equatorial wind waves. *Differential Integral Equations*, 26(3-4):237–252, 2013.

[10] Adrian Constantin, Walter Strauss, and Eugen Vărvărucă. Global bifurcation of steady gravity water waves with critical layers. *Acta Math.*, 217(2):195–262, 2016.

[11] Adrian Constantin and Walter A. Strauss. Exact steady periodic water waves with vorticity. *Comm. Pure Appl. Math.*, 57(4):481–527, 2004.

[12] Adrian Constantin and Walter A. Strauss. Stability properties of steady water waves with vorticity. *Comm. Pure Appl. Math.*, 60(6):911–950, 2007.

[13] Michael G. Crandall and Paul H. Rabinowitz. Bifurcation from simple eigenvalues. *J. Functional Analysis*, 8:321–340, 1971.

[14] M. L. Dubreil-Jacotin. Sur la détermination rigoureuse des ondes permanentes périodiques d'amplitude finie. *Journ. de Math.*, 13:217–289, 1934.

[15] Mark D. Groves. Steady water waves. *J. Nonlinear Math. Phys.*, 11(4):435–460, 2004.

[16] Karl R. Helfrich and W. Kendall Melville. Long nonlinear internal waves. In *Annual review of fluid mechanics. Vol. 38*, pages 395–425. Annual Reviews, Palo Alto, CA, 2006.

[17] David Henry. Large amplitude steady periodic waves for fixed-depth rotational flows. *Comm. Partial Differential Equations*, 38(6):1015–1037, 2013.

[18] David Henry. Steady periodic waves bifurcating for fixed-depth rotational flows. *Quart. Appl. Math.*, 71(3):455–487, 2013.

[19] David Henry. Internal equatorial water waves in the f-plane. *J. Nonlinear Math. Phys.*, 22(4):499–506, 2015.

[20] David Henry and Anca-Voichita Matioc. On the existence of equatorial wind waves. *Nonlinear Anal.*, 101:113–123, 2014.
[21] David Henry and Anca-Voichita Matioc. On the symmetry of steady equatorial wind waves. *Nonlinear Anal. Real World Appl.*, 18:50–56, 2014.
[22] Hung-Chu Hsu. Exact nonlinear internal equatorial waves in the f-plane. *J. Math. Fluid Mech.*, 19(2):367–374, 2017.
[23] Delia Ionescu-Kruse and Calin Iulian Martin. Periodic equatorial water flows from a Hamiltonian perspective. *J. Differential Equations*, 262(8):4451–4474, 2017.
[24] Delia Ionescu-Kruse and Anca-Voichita Matioc. Small-amplitude equatorial water waves with constant vorticity: dispersion relations and particle trajectories. *Discrete Contin. Dyn. Syst.*, 34(8):3045–3060, 2014.
[25] V. Kozlov, N. Kuznetsov, and E. Lokharu. Steady water waves with vorticity: an analysis of the dispersion equation. *J. Fluid Mech.*, 751:R3, 13, 2014.
[26] Vladimir Kozlov and Nikolay Kuznetsov. Dispersion equation for water waves with vorticity and Stokes waves on flows with counter-currents. *Arch. Ration. Mech. Anal.*, 214(3):971–1018, 2014.
[27] Olga A. Ladyzhenskaya and Nina N. Ural'tseva. *Linear and quasilinear elliptic equations.* Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968.
[28] T. Levi-Civita. Determinazione rigorosa delle onde irrotazionali periodiche in acqua profonda. *Rend. Accad. Lincei*, 33:141–150, 1924.
[29] Calin Iulian Martin. On periodic geophysical water flows with discontinuous vorticity in the equatorial f-plane approximation. *Philos. Trans. Roy. Soc. A*, 376(2111):20170096, 23, 2018.
[30] Anca-Voichita Matioc. Steady internal water waves with a critical layer bounded by the wave surface. *J. Nonlinear Math. Phys.*, 19(1):1250008, 21, 2012.
[31] Alexander Mielke. Reduction of quasilinear elliptic equations in cylindrical domains with applications. *Math. Methods Appl. Sci.*, 10(1):51–66, 1988.
[32] John W. Miles. Solitary waves. In *Annual review of fluid mechanics, Vol. 12*, pages 11–43. Annual Reviews, Palo Alto, Calif., 1980.
[33] A. I. Nekrasov. On steady waves. *Izv. Ivanovo-Voznesensk. Politekhn. In-ta*, 3, 1921.
[34] Ronald Quirchmayr. On irrotational flows beneath periodic traveling equatorial waves. *J. Math. Fluid Mech.*, 19(2):283–304, 2017.
[35] Björn Sandstede and Arnd Scheel. Relative Morse indices, Fredholm indices, and group velocities. *Discrete Contin. Dyn. Syst.*, 20(1):139–158, 2008.
[36] Walter A. Strauss. Steady water waves. *Bull. Amer. Math. Soc. (N.S.)*, 47(4):671–694, 2010.
[37] J. F. Toland. Stokes waves. *Topol. Methods Nonlinear Anal.*, 7(1):1–48, 1996.
[38] Erik Wahlén. Steady water waves with a critical layer. *J. Differential Equations*, 246(6):2468–2483, 2009.
[39] Samuel Walsh. Stratified steady periodic water waves. *SIAM J. Math. Anal.*, 41(3):1054–1105, 2009.
[40] Samuel Walsh, Oliver Bühler, and Jalal Shatah. Steady water waves in the presence of wind. *SIAM J. Math. Anal.*, 45(4):2182–2227, 2013.
[41] Ling-Jun Wang. Small-amplitude solitary and generalized solitary traveling waves in a gravity two-layer fluid with vorticity. *Nonlinear Anal.*, 150:159–193, 2017.
[42] Miles H. Wheeler. Simplified models for equatorial waves with vertical structure. *Oceanography*, 30(1):to appear, 2018.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: miles.wheeler@univie.ac.at