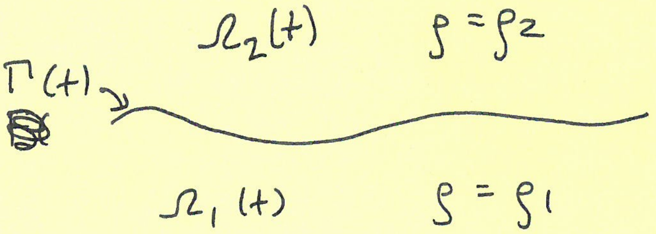
DAY 10

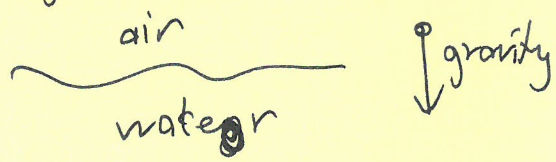
Water waves

So far we have been mostly focused on incompressible fluids with constant density. In the presentations, we will hear more about compressible fluids, but today we will talk about incompressible fluids with variable density.

In particular, we will look at the "simplest" example of two fluid regions with different densities separated by a sharp interface



When the Euler equations are our basic model in each fluid region, this is sometimes called a "water wave problem", because it is a common model for the configuration



which is of course extremely important to many every day applications.

In order to get a closed system, we need to impose boundary conditions on the (time-dependent!) interface $\Gamma(t)$.

Kinematic boundary conditions

Just like with a fixed boundary in Euler, we want particles on $\Gamma(t)$ to stay there for all time. (slip BC). So let $X(t)$ be a particle trajectory and assume that (locally)

$$\Gamma(t) = \{ x : F(x,t) = 0 \}.$$

Then $X(t) \in \Gamma(t)$ $\forall t$ means

$$F(X(t), t) \equiv 0$$

and hence

$$0 = \frac{d}{dt} F(X(t), t) = \frac{DF}{Dt} = F_t + (u \cdot \nabla) F$$

Thus we require

velocity in Ω_1

$$\boxed{\begin{aligned} F_t + (u_1 \cdot \nabla) F &= 0 \quad \text{on } \Gamma(t) \\ F_t + (u_2 \cdot \nabla) F &= 0 \quad \text{on } \Gamma(t) \end{aligned}}$$

velocity in Ω_2

Dynamic boundary condition

We also need to balance the forces acting on the two fluid regions. Assuming that there are no <u>additional</u> surface forces on $\Gamma(t)$ (like, e.g., surface tension), this boils down to assuming:

$$\boxed{p \text{ is continuous across } \Gamma(t)}$$

The full system is then

external forces

$$\rho_2 \frac{Du_2}{Dt} = -\nabla p_2 + f_{ext}$$

$$\rho_1 \frac{Du_1}{Dt} = -\nabla p_1 + f_{ext}$$

$p_1 = p_2$ (dynamic)

$$\left.\begin{aligned} F_t + (u_1\cdot\nabla)F = 0 \\ F_t + (u_2\cdot\nabla)F = 0 \end{aligned}\right\} \text{(kinematic)}$$

Neglecting the upper layer

When only one of the layers is important to us, a common approximation is that the other layer is a region of <u>constant pressure</u>, e.g. atmospheric pressure in the air.

(Can argue based on $\rho_2/\rho_1 \to 0$ for air/water)

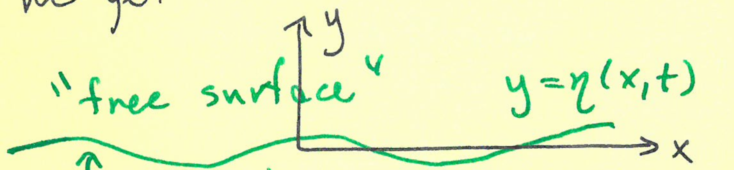
Gravity waves

Take a 2D configuration and neglect the upper layer. Assume that $\Gamma(t)$ is a graph

$$y = \eta(x,t)$$

const. accel due to gravity

and that $f_{ext} = -\rho g e_y$ is gravity. Scaling so that $\rho \equiv 1$ we get

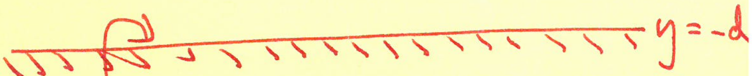


$p = \text{const.},$

$$0 = \frac{D}{Dt}(y-\eta) = v - \eta_t - u\eta_x$$

$$\vec{u} = (u,v)$$

$$\frac{D\vec{u}}{Dt} = -\nabla p - g e_y$$



$v = \vec{u}\cdot\vec{n} = 0$ (rigid body)

Irrotational gravity waves

Typically one further assumes that $\vec{u} = \nabla\phi$ is irrotational. Normalizing ϕ so that <u>Bernoulli's law</u> is

$$\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + p + gy \equiv 0$$

we can then completely eliminate the pressure p from the problem, getting:

"free surface" $y = \eta(x,t)$

$$\text{(WW)} \quad \begin{cases} \eta_t - \phi_y + \phi_x \eta_x = 0 \\ \phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0 \end{cases}$$

$$\Delta \phi = 0$$

"bed" $y = -d$

$\phi_y = 0$

This (and its 3D analogue) is the classical water wave problem.

Despite the many assumptions we have made to get here, it remains an extremely difficult problem:

- the boundary conditions are nonlinear
- the boundary $y = \eta(x,t)$ is itself an unknown in the problem. In this sense it is a free boundary problem.

For this reason (and because it models a system we care a lot about), there are many many many approximation of (WW). Today we will talk about the two most basic ones:

1. Linear waves
2. ~~shallow water waves~~ ~~the shallow water eqns~~

Linear water waves

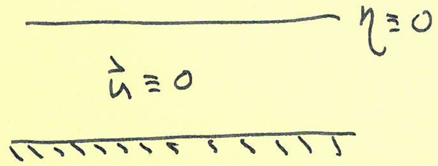
By expanding (WW) in a small parameter $\varepsilon > 0$, we will get an approximate eqn which is

- Linear!
- Posed in a fixed domain.

A boring solution: we easily check that (WW) is solved by

$$\phi \equiv 0, \quad \eta \equiv 0.$$

This represents a flat surface and no motion.



Expansion procedure: Since $\phi \equiv 0$, $\eta \equiv 0$ is a solution, we might hope to introduce a small parameter $\varepsilon > 0$ and expand

$$\phi = \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \varepsilon^3 \phi^{(3)} + \cdots$$

$$\eta = \varepsilon \underbrace{\eta^{(1)}}_{O(1)} + \varepsilon^2 \eta^{(2)} + \varepsilon^3 \eta^{(3)} + \cdots$$

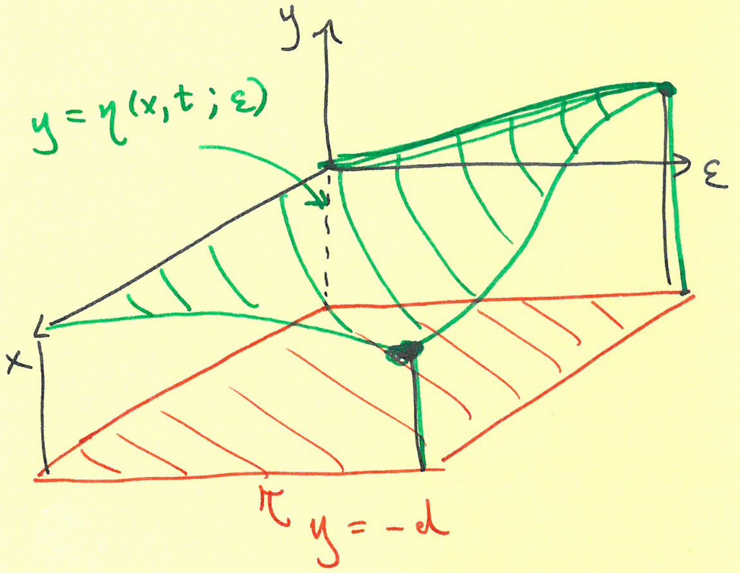
so $\eta = O(\varepsilon)$, i.e. ε is the "amplitude" of the wave.

The fact that the domain on which ϕ is defined depends on η makes this a little more confusing than usual.

I prefer to think of it in the following way:

What we really have is a one-parameter family of solutions $(\phi(\cdot;\varepsilon), \eta(\cdot;\varepsilon))$ of (WW) which are smooth in (x,y,ε) and $\phi(\cdot;0)\equiv 0$, $\eta(\cdot;0)\equiv 0$. Thus ϕ is defined on the set

$$\{(x,y,t,\varepsilon) : -d < y < \eta(x,t;\varepsilon)\}$$



To get approximate eqns for $0 < \varepsilon \ll 1$, we just apply

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$$

to all of the equations.

The easy equations

Suppose that $-d<y<0$. and fix t. Then

$$-d < y < \eta(x,t;\varepsilon)$$

for ε sufficiently small, and so we have

$$0 = \Delta\phi(x,y,t;\varepsilon) \qquad 0 \leq \varepsilon \ll 1.$$

Applying $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$, we get

$$\boxed{\begin{array}{c}\Delta\phi_\varepsilon(x,y,t;0) = 0 \\ \text{for } -d<y<0\end{array}}$$

Similarly, for any (x,t,ε) we have

$$\phi_y(x,-d,t;\varepsilon) = 0$$

and so appling $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$ we get

$$\boxed{\phi_{\varepsilon y}(x,-d,t;0) = 0}$$

The trickier equations

Let's look at the kinematic boundary condition. It says

$$0 = \eta_t(x;\varepsilon) - \phi_y(x,\eta(x;\varepsilon);\varepsilon) + \phi_x(x,\eta(x;\varepsilon);\varepsilon)\,\eta_x(x;\varepsilon).$$

Differentiating using the chain rule we get

$$0 = \eta_{\varepsilon t} - \phi_{\varepsilon y} - \phi_{yy}\eta_\varepsilon + (\phi_{\varepsilon x} + \phi_{xy}\eta_\varepsilon)\eta_x + \phi_x \eta_{\varepsilon x}.$$

At $\varepsilon = 0$, we have $\phi \equiv 0$, $\eta \equiv 0$, and so we just get

$$\boxed{0 = \eta_{\varepsilon t} - \phi_{\varepsilon y} \quad \text{on } y=0,\ \varepsilon = 0 \qquad (1)}$$

That wasn't so bad! Let's do the dynamic boundary condition:

$$0 = \Big(\phi_t + \tfrac{1}{2}\overbrace{|\nabla\phi|^2}^{\phi_x^2+\phi_y^2} + g\eta\Big)\Big|_{y=\eta(x,t;\varepsilon)}$$

Differentiating we get

$$\begin{aligned} 0 = {} & \phi_{\varepsilon t} + \phi_{yt}\eta_\varepsilon \\ & + \phi_x(\phi_{\varepsilon x} + \phi_{xy}\eta_\varepsilon) \\ & + \phi_y(\phi_{\varepsilon y} + \phi_{yy}\eta_\varepsilon) \\ & + g\eta_\varepsilon \end{aligned}$$

and plugging in $\varepsilon = 0$ we are left with

$$\boxed{0 = \phi_{\varepsilon t} + g\eta_\varepsilon \quad \text{on } y=0,\ \varepsilon=0 \quad (2)}$$

~~Taking~~

Now we can combine (1) & (2) to eliminate η:

$$\phi_{\varepsilon tt} \overset{(2)}{=} -g\eta_{\varepsilon t} \overset{(1)}{=} -g(\phi_{\varepsilon y})$$

$$\therefore \quad \boxed{\phi_{\varepsilon tt} + g\phi_{\varepsilon y} = 0}$$

[We can then recover η_ε from (2):

$$\eta_\varepsilon = -\frac{1}{g}\phi_{\varepsilon t}.$$
]

Putting all of this together and DROPPING THE ε's, we get

$\phi_{tt} + g\phi_y = 0$ on $y = 0$

(L) $\Delta\phi = 0$

$\phi_y = 0$ on $y = -d$

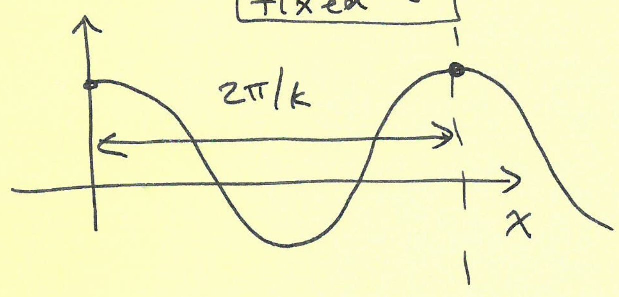
Alright, so what can we learn from this approximation?

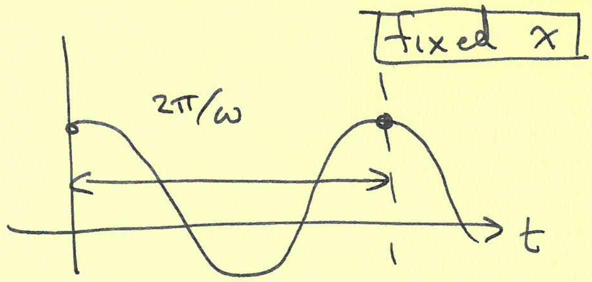
<u>Dispersion relation</u> (L) is a linear problem in a fixed domain with constant coefficients. Moreover, it is translation invariant in both t & x. Thus every solution is a "linear combination" of "plane wave" solutions of the form

$$\phi = \hat{\phi}(y)\, e^{i(kx-\omega t)} \qquad (*)$$

for some constants $k \in \mathbb{R}$ and $\omega \in \mathbb{C}$. It will turn out $\omega \in \mathbb{R}$. Then we can think of

fixed t

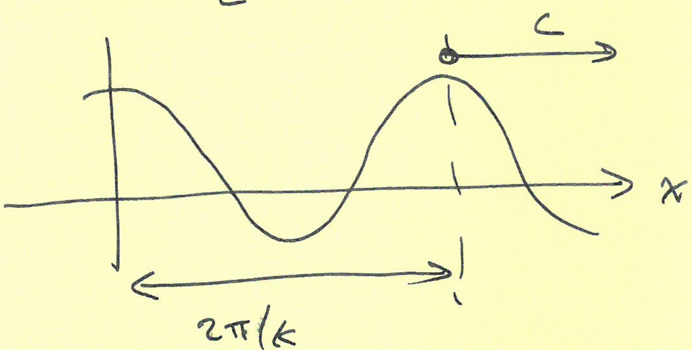




Alternatively, ~~as~~ setting

$$c = \frac{\omega}{k} = \text{"phase velocity"}$$

we have $[kx - \omega t = k(x - ct)]$



We have to use the equation itself, though, to see how ω & k are related.

Plugging ⊛ into (L) we get

$$-\omega^2 \hat{\phi} + g\hat{\phi}_y = 0 \qquad y = 0$$

$$\hat{\phi}_{yy} - k^2 \hat{\phi} = 0$$

$$\hat{\phi}_y = 0 \qquad y = -d$$

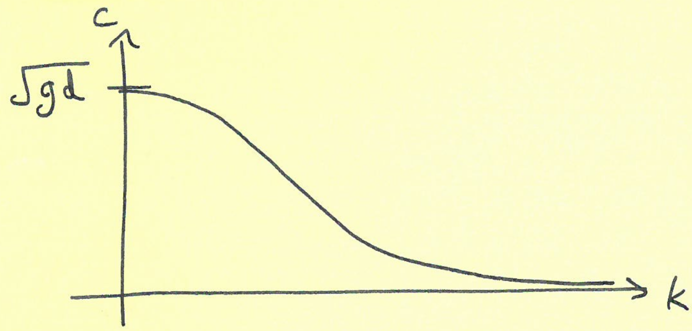
$$\Rightarrow \hat{\phi} = C\cosh[k(y+d)]$$

Plugging into the top BC gives

$$-\omega^2 \cosh(kd) + gk\sinh(kd) = 0$$

or

$$\boxed{\begin{aligned} \omega^2 &= gk\tanh(kd) \\ c^2 &= g\frac{\tanh(kd)}{k} \end{aligned}}$$



This tells us:

- ~~short~~ longer waves are faster than shorter ones
- in the infinite wavelength limit we reach the maximum speed of $\sqrt{gd}$.

This phenomenon where waves with different wavenumbers travel at different speeds is called dispersion.

Group velocity Once you start thinking about more complicated solutions

$$\phi(x,y,t) = \int \hat{\phi}(y,k)\, e^{i(kx - \omega(k)t)}\, dk,$$

dispersion rel'n

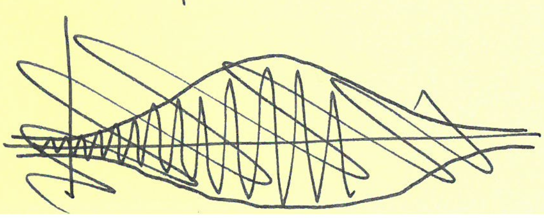
it turns out that the group velocity

$$c_g := \frac{d\omega(k)}{dk}$$

is just if not more important than the phase velocity

$$c_p := \frac{\omega}{k}.$$

c_g is the speed at which energy is transported, and also the speed of wavepackets



13