INTEGRAL AND ASYMPTOTIC PROPERTIES OF SOLITARY WAVES IN DEEP WATER

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Abstract. We consider two- and three-dimensional gravity and gravity-capillary solitary water waves in infinite depth. Assuming algebraic decay rates for the free surface and velocity potential, we show that the velocity potential necessarily behaves like a dipole at infinity and obtain a related asymptotic formula for the free surface. We then prove an identity relating the "dipole moment" to the kinetic energy. This implies that the leading-order terms in the asymptotics are nonvanishing and in particular that the angular momentum is infinite. Lastly we prove a related integral identity which rules out waves of pure elevation or pure depression.

1. Introduction

We consider the motion of an infinitely deep region fluid under the influence of gravity which is bounded above by a free surface, including both *gravity waves*, where the pressure is constant along the free surface, and *gravity-capillary waves*, where it is proportional to the mean curvature. The fluid is assumed to be inviscid and incompressible, and the flow is assumed to be irrotational. We further restrict to traveling-wave solutions which appear steady in a moving reference frame and which are *solitary* in that the free surface approaches some asymptotic height at infinity, normalized to zero.

Solitary waves in finite depth, where the fluid is instead bounded below by a flat bed, have a long and celebrated history; see for instance the reviews [Mil80, DK99, Gro04]. This includes a wide variety of existence results for gravity waves in two dimensions [Lav54, FH54, Bea77, Mie88, AT81] and gravity-capillary waves in two [AK89, Kir88, IK92, BG99, BGT96] and three [GS08, BGSW13, BGW16] dimensions. The two-dimensional gravity waves are *waves of elevation* in that their free surface elevations are everywhere positive [CS88], while some of the gravity-capillary waves are *waves of depression* with negative free surfaces and still others have oscillatory free surfaces which change sign. For the remaining case of three-dimensional gravity waves, Craig [Cra02] has ruled out waves of elevation or depression, while recent results for the time-dependent problem [Wan15b, Wan15a] rule out sufficiently small solitary waves.

In infinite depth, gravity solitary waves are conjectured not to exist, regardless of the dimension. Hur [Hur12] has proved that the only two-dimensional solitary waves whose free surfaces are $O(1/|x|^{1+\varepsilon})$ as $|x| \to \infty$ are *trivial waves* with flat free surfaces. For three-dimensional waves, Craig has shown, as in the finite-depth case, that there

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are no waves of elevation or depression [Cra02]. Also as in the case of finite depth, sufficiently small three-dimensional gravity waves are ruled out by global existence results [Wu11, GMS12] for small data in the time-dependent problem.

Two-dimensional gravity-capillary waves in infinite depth were first rigorously constructed by Iooss and Kirrmann [IK96] following the pioneering numerical work of Longuet-Higgins [LH89] and Vanden-Broeck and Dias [VBD92]. Their proof used normal form techniques; Buffoni [Buf04] and Groves and Wahlén [GW11] have subsequently given variational constructions. One distinguishing feature of these solitary waves is their algebraic decay at infinity. In [IK96] the free surfaces were shown to be $O(1/|x|)$ as $x \to \infty$; Sun [Sun97] later improved this to the expected $O(1/|x|^2)$. More generally, Sun proved that a $O(1/|x|^{1+\varepsilon})$ free surface is automatically $O(1/|x|^2)$, and that in this case several integral identities hold. In particular, the "excess mass" vanishes so that no such wave can be a wave of elevation or depression.

While there are no rigorous constructions of three-dimensional gravity waves in infinite depth, they have been calculated both formally [KA05] and numerically [PVBC05, WM12, AM09]. Interestingly, as the amplitude of these waves approaches zero, their energy approaches a finite value [WM12, Section 3.2.2]. This is consistent with recent global existence results for small data in the time-dependent problem [DIPP16], which rule out solitary waves that are small in a certain function space.

In this paper we simultaneously consider infinite-depth gravity and gravity-capillary solitary waves in dimension $n = 2$ or 3. We assume that the free surface is $O(1/|x|^{n-1+\varepsilon})$ as $|x| \to \infty$ while the velocity potential is $o(1/|x|^{n-2})$. Our first conclusion is that the velocity potential behaves like a dipole near infinity (3), which implies related asymptotics (4) for the free surface. We next give an explicit formula (5) for the kinetic energy in terms of the "dipole moment" and the wave speed. For nontrivial waves, this ensures that the leading-order terms in our asymptotics are nonvanishing, which in turn implies that the angular momentum is infinite (Corollary 3). A modification of the proof of (5) shows that the "excess mass" vanishes (6), ruling out waves of elevation or depression.

We now briefly interpret our results in the context of the previous work mentioned above. The two-dimensional gravity waves we consider are automatically trivial by Hur's nonexistence result [Hur12], so that our results are interesting only in their method of proof. For three-dimensional gravity waves, on the other hand, our results are entirely new. Our nonexistence proof for waves with finite angular momentum complements Craig's nonexistence result [Cra02], which only applies to waves of elevation and depression, as well as the time-dependent results [Wu11, GMS12], which only rule out waves which are sufficiently small. For two-dimensional gravity-capillary waves, we improve upon Sun's asymptotic bounds [Sun97] by proving asymptotic formulas for both the free surface and velocity potential, with nonvanishing leading-order terms. Given this dipole-like behavior of the velocity potential, somewhat formal proofs of our integral identities were given in this case by Longuet-Higgens [LH89]. To our knowledge, the only rigorous results for three-dimensional solitary capillary-gravity waves in infinite depth are Hur's recent generalization [Hur15] of one of Sun's two-dimensional identities and the implicit nonexistence result for small waves in [DIPP16].

The two-dimensional results [Sun97, Hur12] both use conformal mappings to obtain a problem in a fixed domain. Hur goes on to write a non-local Babenko-type equation for the free surface elevation as a function of the velocity potential, while Sun exploits the existence of explicit Greens functions. In three dimensions, these arguments break down entirely. Instead of conformal mappings, we use the Kelvin transform. This does not fix the domain, but does convert questions about the asymptotic behavior of the velocity potential near infinity into questions about the regularity of the transformed potential near a finite point on the boundary of the transformed fluid domain. We obtain this boundary regularity using standard Schauder estimates for weak solutions to elliptic equations.

This paper is organized as follows. In Section 2, we state our main results. In Section 3, we prove Theorem 1 on the asymptotic behavior of the velocity potential and free surface near infinity. To streamline the presentation, some of the more technical details are deferred to Appendix A. In Section 4, we prove Theorem 2, which expresses the kinetic energy in terms of the wave speed and dipole moment, as well as its corollaries. The proof involves applying the divergence theorem to a carefully chosen vector field and then using the asymptotics from Theorem 1 to deal with one of the boundary terms.

2. Results

There are two distinguished reference frames for a solitary wave: a moving frame where the motion appears steady, and another “lab” frame where the fluid velocity is assumed to vanish at infinity. As is common practice, we measure positions in the first frame and velocities in the second. We set $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $n = 2$ or 3, with x' the horizontal coordinate and x_n the vertical coordinate. Assuming that the free surface S is a graph $x_n = \eta(x')$, the semi-infinite fluid domain is

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : x_n < \eta(x')\}.$$

Since the fluid is irrotational, the fluid velocity in the lab frame is the gradient of a velocity potential φ. The equations satisfied by φ and η are

$$\Delta\varphi = 0 \qquad \text{in } \Omega, \tag{1a}$$

$$\nabla\varphi \cdot N = c \cdot N \qquad \text{on } S, \tag{1b}$$

$$\tfrac{1}{2}|\nabla\varphi|^2 - c \cdot \nabla\varphi + g\eta = -\sigma\nabla \cdot N \qquad \text{on } S, \tag{1c}$$

$$\eta \to 0 \qquad \text{as } |x'| \to \infty, \tag{1d}$$

$$\varphi, \nabla\varphi \to 0 \qquad \text{as } |x| \to \infty, \tag{1e}$$

where here $g > 0$ is the constant acceleration due to gravity, $\sigma \geq 0$ is the constant coefficient of surface tension, $c = (c', 0) \in \mathbb{R}^n$ is the (nonzero) wavespeed, and $N = N(x')$ is the unit normal vector to S pointing out of Ω.

Our main assumptions are that φ satisfies

$$\varphi = o\left(\frac{1}{|x|^{n-2}}\right) \qquad \text{as } |x| \to \infty, \tag{2a}$$

while η and its derivatives satisfy

$$\eta = O\left(\frac{1}{|x'|^{n-1+\varepsilon}}\right), \quad \frac{\partial \eta}{\partial x'_i} = O\left(\frac{1}{|x'|^{n+\varepsilon}}\right), \quad \frac{\partial^2 \eta}{\partial x'_i \partial x'_j} = O\left(\frac{1}{|x'|^{n+1+\varepsilon}}\right), \tag{2b}$$

as $|x'| \to \infty$ for some $\varepsilon \in (0,1)$ and all i,j. Note that (2a) follows from (1e) when $n=2$.

Our first result is that φ behaves like a dipole at infinity.

Theorem 1. *Let $\eta \in C^2(\mathbb{R}^{n-1})$ and $\varphi \in C^2(\overline{\Omega})$ solve (1) with $\sigma \geq 0$ and suppose that the decay estimates (2) hold. Then there exists a "dipole moment" $p = (p', 0) \in \mathbb{R}^n$ such that φ satisfies*

$$\varphi = \frac{p \cdot x}{|x|^n} + O\left(\frac{1}{|x|^{n-1+\varepsilon}}\right), \ \nabla\varphi = \nabla\frac{p \cdot x}{|x|^n} + O\left(\frac{1}{|x|^{n+\varepsilon}}\right) \quad \text{as } |x| \to \infty \tag{3}$$

while η satisfies

$$\eta = \frac{1}{g|x'|^n}\left(c \cdot p - n\frac{(c\cdot x')(p \cdot x')}{|x'|^2}\right) + O\left(\frac{1}{|x'|^{n+\varepsilon}}\right) \quad \text{as } |x'| \to \infty. \tag{4}$$

Dipole asymptotics along the lines of (3) feature in Longuet-Higgins's numerical calculations of two-dimensional gravity-capillary waves [LH89, Section 6] as well as Benjamin and Olver's discussion of conserved quantities in the two-dimensional time-dependent problem in [BO82, Section 6.5]. Comparing to small-amplitude expansions, the asymptotic formula (4) is consistent with (4.1) in [ADG98] in two dimensions and (5.6) in [KA05] in three dimensions when p and c are parallel.

For two-dimensional gravity-capillary waves, Sun proves, roughly, that the decay $\eta = O(|x'|^{-1-\varepsilon})$ forces $\eta = O(|x'|^{-2})$ [Sun97]. Our result is stronger in that it identifies the leading order term in the asymptotics, but weaker in that it requires slightly more information about derivatives of η (Sun works in weighted $C^{1+\alpha}$ spaces). Sun also allows for a semi-infinite upper layer with a different density, and, in the important special case when $\sigma > |c|^2/4g$, only needs to assume $\eta = O(|x'|^{-\varepsilon})$. As mentioned at the end of Section 1, his proof uses conformal mappings and hence does not generalize to three dimensions.

Our next result is an integral identity involving the dipole moment p from Theorem 1.

Theorem 2. *In the setting of Theorem 1, the kinetic energy, dipole moment p, and wave speed c are related by*

$$\frac{1}{2}\int_\Omega |\nabla\varphi|^2\, dx = -\frac{\pi^{n/2}}{2\Gamma(\frac{n}{2})}(c \cdot p). \tag{5}$$

For two-dimensional gravity-capillary waves, Longuet-Higgins [LH89] gave a formal proof of (5) assuming (3); also see equation (6.22) in [BO82]. For three-dimensional waves, however, (5) seems to be new.

Theorem 2 implies $c \cdot p < 0$ for nontrivial waves with $\varphi \not\equiv 0$, and hence that the leading-order terms in (3) and (4) do not vanish. For $n=2$, solving (5) for p and substituting into (4) yields

$$\eta = \left(\frac{\pi}{g}\int_\Omega |\nabla\varphi|^2\,dx\right)\frac{1}{|x'|^2} + O\left(\frac{1}{|x'|^{2+\varepsilon}}\right),$$

which for instance implies that $\eta > 0$ for $|x'|$ sufficiently large. For $n = 3$, η instead takes both positive and negative values in any neighborhood of infinity.

Another consequence of Theorem 2 is the following dichotomy.

Corollary 3. *In the setting of Theorem 1, either the angular momentum* $\int_\Omega x\times\nabla\varphi\,dx$ *is infinite or the wave is trivial, i.e.* $\varphi \equiv 0$ *and* $\eta \equiv 0$.

For two-dimensional time-dependent waves, Benjamin and Olver observe that $p' \neq 0$ causes the integral defining the total angular momentum to diverge [BO82, Section 6.5]. This motivates them to impose additional restrictions guaranteeing $p' = 0$, but not necessarily $p_n = 0$. For two-dimensional gravity-capillary waves, Longuet-Higgins [LH89] explains how dipole behavior for the velocity potential causes the horizontal momentum to be indeterminant in that, for instance, $\nabla\varphi \notin L^1(\Omega)$.

A final corollary of the proof of Theorem 2 is that the "excess mass" vanishes.

Corollary 4. *In the setting of Theorem 1, the wave has zero excess mass in that*

$$\int_{\mathbb{R}^{n-1}} \eta(x')\,dx' = 0. \tag{6}$$

For two-dimensional capillary-gravity waves, (6) was derived in [LH89], and a stronger version of Corollary 4 was proved rigorously in [Sun97]. For three-dimensional waves, however, Corollary 4 appears to be new. An obvious consequence is that no nontrivial waves satisfying (2) are waves of elevation (with $\eta > 0$) or depression (with $\eta < 0$). In this setting, however, waves of elevation and depression have been ruled out by Craig [Cra02] using maximum principle arguments, without imposing any assumptions on the decay rates of η and φ.

3. Proof of Theorem 1

The main ingredient in the proof of Theorem 1 is the following lemma, which states that (3) holds independently of the dynamic boundary condition (1c).

Lemma 5. *Let* $\varphi \in C^2(\overline{\Omega})$ *and* $\eta \in C^2(\mathbb{R}^{n-1})$ *solve* (1a)–(1b)*. If the decay estimates* (2) *hold, then there exists* $p = (p', 0) \in \mathbb{R}^n$ *(possibly zero) so that* φ *satisfies the asymptotic conditions* (3)*.*

Proof. We apply the Kelvin transform, setting

$$\tilde{x} = T(x) = \frac{x}{|x|^2}, \qquad \tilde{\varphi}(\tilde{x}) = \frac{1}{|\tilde{x}|^{n-2}}\varphi\left(\frac{\tilde{x}}{|\tilde{x}|^2}\right), \qquad \Omega^\sim = T(\Omega \setminus B_1),$$

where $B_1 = \{x : |x| < 1\}$ is an open ball centered at the origin. Note that $T(T(x)) = x$. This change of variables converts asymptotic questions about φ as $|x| \to \infty$ into local questions about $\tilde{\varphi}$ in a neighborhood of $0 \in \partial\Omega^\sim$. For instance, (2a) implies that $\tilde{\varphi}$ extends to a $C^0(\overline{\Omega^\sim})$ function with $\tilde{\varphi}(0) = 0$.

Using the decay assumptions (2), we show in the appendix that $\Omega^\sim$ has a C^2 boundary portion $S^\sim$ containing 0 and that $\tilde\varphi \in H^1(\Omega^\sim)$ is a weak solution to the boundary-value problem

$$\Delta\tilde\varphi = 0 \text{ in } \Omega^\sim, \qquad \frac{\partial\tilde\varphi}{\partial\tilde N} + \alpha\tilde\varphi = \beta \text{ on } S^\sim, \tag{7}$$

where $\alpha, \beta \in C^\varepsilon(S^\sim)$ are given up to a sign by

$$\alpha(\tilde x) = -(n-2)\big(x\cdot N(x)\big), \quad \beta(\tilde x) = |x|^n\big(c\cdot N(x)\big) \qquad \text{on } S^\sim \setminus \{0\}$$

and $\alpha(0) = \beta(0) = 0$. Standard elliptic regularity theory (for instance Theorem 5.51 in [Lie13]) then implies that $\tilde\varphi \in C^{1+\varepsilon}(\Omega^\sim \cup S^\sim)$. In particular, setting $p = (p', p_n) = \nabla\tilde\varphi(0)$, we have an expansion

$$\tilde\varphi(\tilde x) = p\cdot\tilde x + O(|\tilde x|^{1+\varepsilon}), \qquad \nabla\tilde\varphi(\tilde x) = p + O(|\tilde x|^\varepsilon) \tag{8}$$

as $\tilde x \to 0$. Rewriting (8) in terms of φ and $\nabla\varphi$ then yields (3) as desired. Finally, plugging $\tilde x = 0$ in the boundary condition in (7), we find

$$p_n = \frac{\partial\tilde\varphi}{\partial\tilde N}(0) = -\alpha(0)\tilde\varphi(0) + \beta(0) = 0. \qquad \square$$

Theorem 1 now follows from Lemma 5 and the dynamic boundary condition (1c).

Proof of Theorem 1. We have already shown (3), so it suffices to prove (4). From (2b), we have $\nabla\cdot N = O(1/|x'|^{n+1+\varepsilon})$. Solving the dynamic boundary condition (1c) for η and plugging in (3) therefore yields

$$\eta(x') = \frac{1}{g|x|^n}\left(c\cdot p - n\frac{(c\cdot x')(p\cdot x')}{|x|^2}\right) + O\left(\frac{1}{|x|^{n+\varepsilon}}\right) + O\left(\frac{1}{|x'|^{n+1+\varepsilon}}\right), \tag{9}$$

where here x is shorthand for $(x', \eta(x'))$. Since $\eta \to 0$ as $|x'| \to \infty$, we can replace each occurrence of x in (9) with $(x', 0)$, yielding (4) as desired. $\square$

4. Integral identities

Let $B_r = \{x : |x| < r\}$ denote the open ball with radius r centered at the origin, and let $e_n = (0, 1)$ be the unit vector in the vertical direction.

Proof of Theorem 2. Consider the vector field

$$\begin{aligned} A := &\left(-\frac{|c|^2}{g}(e_n\cdot\nabla\varphi) + c\cdot x + \varphi\right)\nabla\varphi \\ &+ \frac{|c|^2}{g}\left(\frac{1}{2}|\nabla\varphi|^2 - c\cdot\nabla\varphi\right)e_n + \left(\frac{|c|^2}{g}(e_n\cdot\nabla\varphi) - \varphi\right)c. \end{aligned}$$

A simple calculation using only the fact that φ is harmonic shows that $\nabla\cdot A = |\nabla\varphi|^2$. Thus we can apply the divergence theorem to A on the bounded region $B_r \cap \Omega$ to obtain

$$\int_{B_r\cap\Omega} |\nabla\varphi|^2\,dx = \int_{B_r\cap S} A\cdot N\,dS + \int_{\partial B_r\cap\Omega} A\cdot N\,dS. \tag{10}$$

Note that $B_r \cap S$ is the portion of the boundary of $B_r \cap \Omega$ on the free surface while $\partial B_r \cap \Omega$ is the portion inside the fluid.

On the free surface S, we have

$$N = \frac{(-\nabla\eta, 1)}{\sqrt{1+|\nabla\eta|^2}}, \qquad dS = \sqrt{1+|\nabla\eta|^2}\, dx', \tag{11}$$

while the boundary conditions (1b) and (1c) imply

$$A\cdot N = (c\cdot x)(c\cdot N) - \left(\frac{|c|^2\sigma}{g}\nabla\cdot N + |c|^2\eta\right)(e_n\cdot N).$$

Thus the first term on the right hand side of (10) can be rewritten as

$$\begin{aligned}\int_{B_r\cap S} A\cdot N\, dS &= -\int_{B_r\cap S}\left((c\cdot x)(c\cdot\nabla\eta) + \frac{|c|^2\sigma}{g}\nabla\cdot N + |c|^2\eta\right)dx' \\ &= -\int_{B_r\cap S}\nabla\cdot\left(\frac{|c|^2\sigma}{g}N + \eta(c\cdot x)c\right)dx' \\ &= \int_{\partial B_r\cap S}\left(\frac{|c|^2\sigma}{g}N + \eta(c\cdot x)c\right)\cdot\nu'\, ds, \end{aligned}\tag{12}$$

where here the outward-pointing normal $\nu'\colon T\to\mathbb{R}^{n-1}$ and measure ds are with respect to the projection of $\partial B_r\cap S$ onto $\mathbb{R}^{n-1}$.

Plugging (12) into (10) we obtain

$$\begin{aligned}\int_{\Omega\cap B_r}|\nabla\varphi|^2\,dx = \frac{|c|^2\sigma}{g}\int_{S\cap\partial B_r} N\cdot\nu'\,ds - \int_{S\cap\partial B_r}\eta(c\cdot x)(c\cdot\nu')\,ds \\ + \int_{\Omega\cap\partial B_r} A\cdot N\,dS.\end{aligned}\tag{13}$$

From (11) and (2b) we see that the first integrand on the right hand side of (13) is $O(|\nabla\eta|) = O(|x'|^{-(n+\varepsilon)})$ while the second integrand is $O(|x'|^{-(n+2+\varepsilon)})$. Thus these first two integrals vanish as $r\to\infty$. Thanks to the asymptotic conditions (3) proved in Theorem 1, the remaining integral converges, as $r\to\infty$, to the constant value

$$\begin{aligned}\int_{\partial B_r\cap\{x_n<0\}}\left((c\cdot x)\nabla\frac{p\cdot x}{|x|^n} - \frac{p\cdot x}{|x|^n}c\right)\cdot\frac{x}{|x|}\,dS &= -n\int_{\partial B_1\cap\{x_n<0\}}(c\cdot x)(p\cdot x)\,dS \\ &= -\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})}(c\cdot p),\end{aligned}$$

leaving us with (5) as desired. □

Proof of Corollary 4. We follow the proof of Theorem 2, but with A replaced by the vector field

$$\tilde{A} = -\frac{|c|^2}{g}(e_n\cdot\nabla\varphi)\nabla\varphi + \frac{|c|^2}{g}\left(\frac{1}{2}|\nabla\varphi|^2 - c\cdot\nabla\varphi\right)e_n + \frac{|c|^2}{g}(e_n\cdot\nabla\varphi)c$$

obtained by dropping all of the terms in A without a factor of $|c|^2/g$. A simple calculation shows that $\nabla\cdot\tilde{A} = 0$, and the boundary conditions (1b) and (1c) give

$$\tilde{A}\cdot N = -|c|^2\eta - \frac{|c|^2\sigma}{g}\nabla\cdot N$$

on the free surface S. As in the proof of Theorem 2, we apply the divergence theorem, first to $\tilde{A}$ in $B_r \cap \Omega$, and then again on $S \cap B_r$, obtaining

$$|c|^2 \int_{B_r \cap S} \eta \, dx' = -\frac{|c|^2 \sigma}{g} \int_{S \cap \partial B_r} N \cdot \nu' \, ds + \int_{\Omega \cap \partial B_r} \tilde{A} \cdot N \, dS. \tag{14}$$

The first term on the right hand side of (14) vanishes as $r \to \infty$ as in proof of Theorem 2. The second term vanishes since $\tilde{A} = O(|\nabla \varphi|) = O(|x|^{-n})$ by (3). From (2b) we know that $\eta \in L^1(\mathbb{R}^{n-1})$, so taking $r \to \infty$ in (14) yields (6) as desired. $\square$

Proof of Corollary 3. For any $r > 0$, the asymptotic condition (3) implies that the integral

$$\int_{\Omega \cap \partial B_r} x \times \nabla \varphi \, dS$$

converges, as $r \to \infty$, to the constant value

$$\int_{\partial B_r \cap \{x_n < 0\}} x \times \nabla \frac{p \cdot x}{|x|^n} \, dS = -p \times \int_{\partial B_1 \cap \{x_n < 0\}} x \, dS = \frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} p \times e_n. \tag{15}$$

Suppose that the angular momentum is finite. Then the right hand side of (15) must be zero, which forces $p \times e_n = 0$ and hence $p' = 0$. But then $c \cdot p = 0$ so that (5) gives $\nabla \varphi \equiv 0$ and therefore $\varphi \equiv 0$.

It remains to show $\eta \equiv 0$. Plugging $\varphi \equiv 0$ into (1c), we find $g\eta = -\sigma \nabla \cdot N$. At a positive maximum of η, this reduces to $0 < g\eta = \sigma \Delta \eta \leq 0$, a contradiction. Similarly at a negative minimum of η we have $0 > g\eta = \sigma \Delta \eta \geq 0$, again a contradiction, and we conclude that $\eta \equiv 0$. $\square$

Appendix A.

In this appendix we provide the remaining details in the proof of Lemma 5.

Setting $S^\sim = (B_\delta \cap T(S)) \cup \{0\} \subset \partial \Omega^\sim$ for δ sufficiently small, we first claim that $S^\sim$ is a C^2 graph $\tilde{x}_n = f(\tilde{x}')$, which will imply that $S^\sim$ is a C^2 boundary portion. As an intermediate step, we define yet another variable

$$x^* = \frac{x'}{|x'|^2}$$

which, on $S^\sim \setminus \{0\}$, is related to $\tilde{x}'$ via

$$\tilde{x}' = \frac{x^*}{1 + |x^*|^2 \eta^2(x^*/|x^*|^2)}. \tag{16}$$

Our decay assumptions (2b) easily imply that $|x^*|^2 \eta^2(x^*/|x^*|^2)$ extends to a C^2 function of x^* in a neighborhood of $x^* = 0$ which vanishes at $x^* = 0$ together with its first and second derivatives. Thus we can use the implicit function theorem to solve (16) for x^* as a C^2 function of $\tilde{x}'$. Expressing $\tilde{x}_n$ in terms of x^*,

$$\tilde{x}_n = \frac{|x^*|^2 \eta(x^*/|x^*|^2)}{1 + |x^*|^2 \eta^2(x^*/|x^*|^2)}, \tag{17}$$

(2b) similarly guarantees that $\tilde{x}_n$ can be extended to a C^2 function of x^* in a neighborhood of $x^*=0$. Composing the C^2 mappings $x^* \mapsto \tilde{x}_n$ and $\tilde{x}' \mapsto x^*$ yields the desired equation $\tilde{x}_n = f(\tilde{x}')$ for $S^\sim$.

We now consider the boundary condition satisfied by $\tilde{\varphi}$ on $S^\sim$. A calculation shows that a C^1 unit normal $\tilde{N}$ on $S^\sim$ is given in terms of the normal vector N on S via the formula

$$\tilde{N}(\tilde{x}) = N(x) - 2\frac{N(x)\cdot x}{|x|^2}.$$

For simplicity assume that $\tilde{N}$ points out of $\Omega^\sim$; otherwise the definitions of β and α below are off by an unimportant sign. Differentiating the identity

$$\varphi(x) = \frac{1}{|x|^{n-2}}\tilde{\varphi}\left(\frac{x}{|x|^2}\right)$$

and using the boundary condition (1b), we find that, on $S^\sim \setminus \{0\}$,

$$c\cdot N = \frac{\partial\varphi}{\partial N} = -(n-2)\frac{x\cdot N}{|x|^n}\tilde{\varphi} + \frac{1}{|x|^n}\frac{\partial\tilde{\varphi}}{\partial\tilde{N}}.$$

Multiplying through by $|x|^n$, we write this as

$$\frac{\partial\tilde{\varphi}}{\partial\tilde{N}} + \alpha\tilde{\varphi} = \beta \qquad \text{on } S^\sim \setminus \{0\},$$

where α and β are defined on $S^\sim \setminus \{0\}$ by

$$\alpha(\tilde{x}) = -(n-2)(x\cdot N), \qquad \beta(\tilde{x}) = |x|^n(c\cdot N).$$

Clearly $\alpha, \beta \in C^1(S^\sim \setminus \{0\})$. Using our decay assumptions (2b), we check that they extend to C^ε functions of $\tilde{x}'$ vanishing at $\tilde{x}'=0$. We remark that only this last extension of β requires the full force of (2b); the other extensions only require $\eta = O(1/|x|^\varepsilon)$, $D\eta = O(1/|x|^{1+\varepsilon})$, and $D^2\eta = O(1/|x|^{2+\varepsilon})$.

Next we show that $\tilde{\varphi} \in H^1(\Omega^\sim)$. From (2b) we know $\tilde{\varphi} \in C^0(\overline{\Omega^\sim}) \cap C^2(\overline{\Omega^\sim} \setminus \{0\})$, so it is enough to show $\nabla\tilde{\varphi} \in L^2(\Omega^\sim \cap B_R)$ for some $R>0$. Fix R small enough that $B_{2R} \cap \partial\Omega^\sim \subset S^\sim$, let $\eta_0 \in C_c^\infty(\mathbb{R})$ be a nonnegative function satisfying $\eta_0(s)=0$ for $s<1$ and $\eta_0(s)=1$ for $s>2$, and for $r<R/2$ define

$$\eta(\tilde{x};r) = \eta_0(r^{-1}|\tilde{x}|)\Big(1-\eta_0\big(R^{-1}|\tilde{x}|\big)\Big).$$

Multiplying $\Delta\tilde{\varphi}=0$ by $\eta^2\tilde{\varphi}$ and integrating by parts, we find

$$\begin{aligned}\int_{\Omega^\sim} |\eta\nabla\tilde{\varphi}|^2\,d\tilde{x} &= -2\int_{\Omega^\sim}\eta\tilde{\varphi}\nabla\eta\cdot\nabla\tilde{\varphi}\,d\tilde{x} + \int_{S^\sim}\eta\tilde{\varphi}(\beta-\alpha\tilde{\varphi})\,dS \\ &\le C\left(1+\|\nabla\eta\|_{L^2(\mathbb{R}^n)}\|\eta\nabla\tilde{\varphi}\|_{L^2(\Omega^\sim)}\right),\end{aligned} \tag{18}$$

where C depends on the L^∞ norms of $\tilde{\varphi}, \alpha, \beta$. Since

$$\|\nabla\eta\|_{L^2(\mathbb{R}^n)} \le C(1+r^{n-2}) \le C,$$

(18) implies an upper bound on $\|\eta\nabla\tilde{\varphi}\|_{L^2(\Omega^\sim)}$ independent of $r<R/2$. Sending $r\to 0$, we obtain $\nabla\tilde{\varphi} \in L^2(\Omega^\sim \cap B_R)$ as desired.

Finally, we claim that $\tilde{\varphi}$ is a weak solution to (7). Certainly (after perhaps changing the definitions of α, β by a sign)

$$\int_{\Omega^\sim} \nabla\tilde{\varphi} \cdot \nabla v \, d\tilde{x} = \int_{S^\sim} (\alpha\tilde{\varphi} - \beta) v \, d\tilde{x} \tag{19}$$

for all smooth $v \in H^1(\Omega^\sim)$ vanishing in a neighborhood of 0. Such v are dense in $H^1(\Omega^\sim)$ (see, for instance, Lemmas 17.2 and 17.3 in [Tar07]). Since $\tilde{\varphi} \in H^1(\Omega^\sim)$, (19) therefore holds for all $v \in H^1(\Omega^\sim)$ and the claim is proved.

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