LOCAL AND GLOBAL BIFURCATION IN FLUID MECHANICS

MILES H. WHEELER

1. Motivation

1.1. **Fluid mechanics.** Obviously fluids are hugely important in every day life: tea in a mug, plumbing, the oceans, the atmosphere, rivers, the interior of the earth, blood in our veins. On the other hand, fluid mechanics isn't exactly cutting-edge theoretical physics anymore, and most authors publishing in fluid mechanics journals are engineers or mathematicians. Many of the basic mathematical questions about the PDEs of fluid mechanics remain open, even if they are "closed" for the physicists.

Perhaps the most famous equations are the **Navier–Stokes equations**:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u,$$
$$\nabla \cdot u = 0.$$

Here $u(x,y,z,t)$ is the velocity field, $p(x,y,z,t)$ is the pressure, and $\nu \geq 0$ is a constant related to frictional (viscous) forces in the fluid. When $\nu = 0$ we obtain the **Euler equations**.

One of the Clay Math Millennium problems (1 million dollar prize) is: *Do the Navier Stokes equations in 3D have smooth initial solutions for smooth initial data, or do the solutions break down?* The same question is also open for for the Euler equations!

1.2. **Model equations and special solutions.** The basic equations of fluid mechanics are complicated and have few explicit solutions. As usual with PDEs, we can get more understanding by deriving approximate models in asymptotic regimes, looking for special solutions like traveling waves, or doing both at the same time!

Rigorously justifying this sort of procedure from a mathematical point of view, though, can be highly nontrivial. In this lecture we will consider the question: *Does a traveling-wave solution of an approximate model approximate an exact traveling-wave solution of the full equations? (Or just a solution that is approximately traveling?)*

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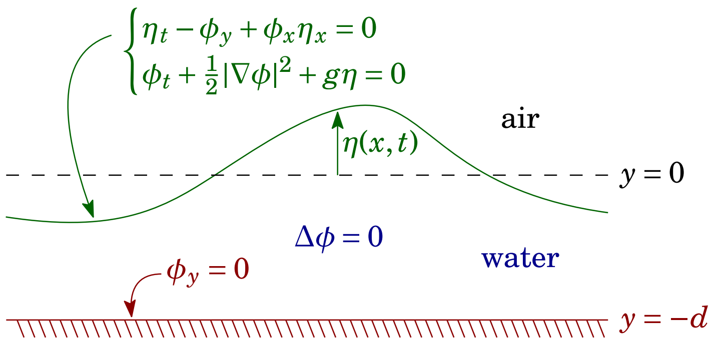
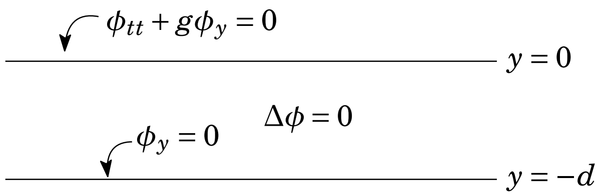
1.3. **Stokes waves.** Consider the following two-dimensional problem:



Here $u = \nabla\varphi$ solves the Euler equations in the water region, and the pressure is constant on the surface. Moreover, fluid particles on the surface and bed remain there for all time.

Notice that $\eta = 0$ and $\varphi = 0$ is always a solution, describing a flat surface and no motion. When η and φ are both "small" then a formal calculation leads to the approximate ("linearized") equation

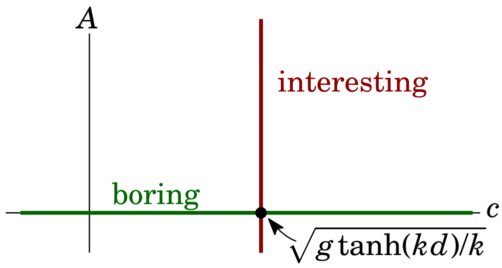


At this level of approximation, the surface profile $\eta(x,t) = -\frac{1}{g}\varphi_t(x,0,t)$.

Looking for traveling wave solutions that depend on $(x - ct, y)$, we eventually get

$$\eta = A\cos x, \qquad \varphi = -gA\cosh(k(y+d))\cos x, \qquad c^2 = g\frac{\tanh(kd)}{k}$$

where the amplitude A is a free constant. The last equation is called the **dispersion relation**. Fixing k, the solution-set can be represented by two crossed lines in the (A, c) plane:



The *mathematical* question is to what extent this picture applied to the *full* problem without any approximations. What does the amplitude-wavespeed diagram look like, for instance?

2. Local bifurcation theory

Recall the following version of the implicit function theorem:

Theorem 1 (Implicit function theorem)**.** *If $F: X \times \Lambda \to Y$ is a C^k mapping between Banach spaces for some $k \geq 1$, $F(x_0, \lambda_0) = 0$, and the partial derivative $F_x(x_0, \lambda_0)$ is invertible, then we can "locally solve $F(x, \lambda) = 0$ for x as a function of λ".*

This describes situations where the zero-set of F looks like a smooth curve or surface. Since we want something more like two intersecting lines, we have to look at situations where the hypotheses of the implicit function theorem fail. This is (local) bifurcation theory!

2.1. **Simplest example.**

Let $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be smooth, and suppose that $F(0, \lambda) = 0$ for all λ. These are the boring or **trivial** solutions that we already know about. If $F_x(0, \lambda_0) \neq 0$, then by the implicit function theorem there are no other solutions near $(0, \lambda_0)$, so let's assume from now on that $F_x(0, \lambda_0) = 0$. (This is like the dispersion relation in our model problem.)

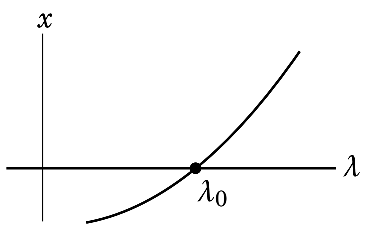
To see what's going on in this case, we Taylor expand

$$\begin{aligned} 0 = F(x, \lambda) &= \cancel{F(0, \lambda_0)} + \cancel{F_x(0, \lambda_0)}x + \cancel{F_\lambda(0, \lambda_0)}x \\ &\quad + \frac{1}{2}F_{xx}(0, \lambda_0)x^2 + F_{x\lambda}(0, \lambda_0)x\lambda + \frac{1}{2}\cancel{F_{\lambda\lambda}(0, \lambda_0)}\lambda^2 + \cdots \end{aligned}$$

Dividing through by x, we therefore hope there is a nontrivial solution given by

$$\lambda - \lambda_0 = -\frac{F_{xx}(0, \lambda_0)}{2F_{x\lambda}(0, \lambda_0)}x + \cdots,$$

at least if $F_{x\lambda}(0, \lambda_0) \neq 0$.



To prove this, we use the fundamental theorem of calculus and the fact that $F(0, \lambda) = 0$ to write

$$F(x, \lambda) = \int_0^1 \frac{d}{dt} F(tx, \lambda)\, dt = \Big(\int_0^1 F_x(tx, \lambda)\, dt\Big)x.$$

Thus, for $x \neq 0$, $F(x, \lambda) = 0$ if and only if $G(x, \lambda) = 0$ where

$$G(x, \lambda) = \int_0^1 F_x(tx, \lambda)\, dt = \frac{F(x, \lambda)}{x}.$$

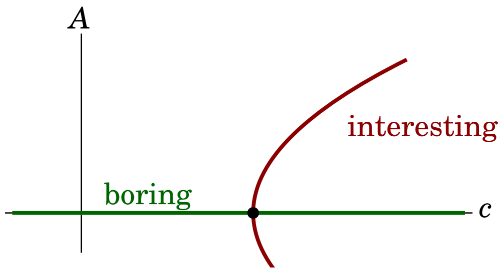
Now we simply apply the implicit function theorem to G. We calculate

$$G(0,\lambda_0) = \int_0^1 F_x(0,\lambda_0)\,dt = F_x(0,\lambda_0) = 0,$$
$$G_\lambda(0,\lambda_0) = \int_0^1 F_{x\lambda}(0,\lambda_0)\,dt = F_{x\lambda}(0,\lambda_0) \neq 0.$$

Thus we can *uniquely* locally solve $G(x,\lambda) = 0$ for λ as a function of x.

2.2. **Generalization to higher dimension.** As often happens in mathematics, the above argument can be generalized in many directions. When F is a mapping between Banach spaces, it is the famous Crandal–Rabinowitz theorem (1971).

This theorem can be applied in our water wave problem above to justify our formal linear calculations. These waves are called **Stokes waves**. With a bit more calculation (third derivatives of F, basically), you can start to figure out what the amplitude-wavespeed diagram looks like:



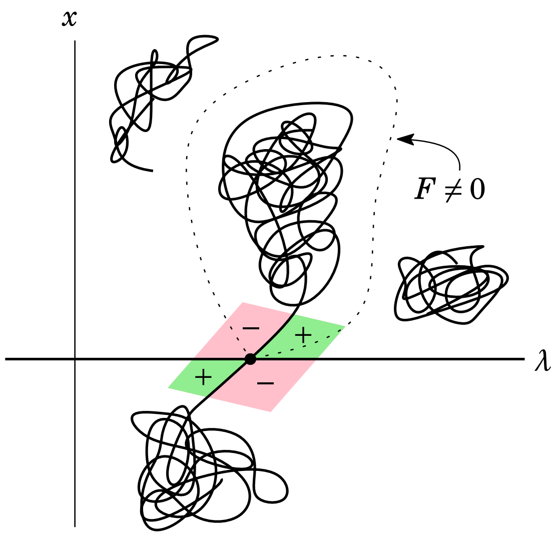
3. Global bifurcation

Let's think about a slightly different question: What is the *global* structure of our solution set?

3.1. **Simplest example.** For $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as above, we can provide a pictorial "proof" of the following claim.

Claim 1. *The solutions we found with $x > 0$ are part of a connected component S of nontrivial solutions which is either unbounded or approaches another trivial solution $(0,\lambda_1) \neq (0,\lambda_0)$.*

If not, then we have something like the following diagram:



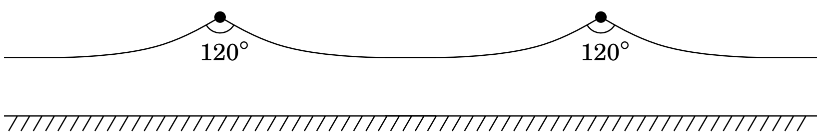
Near the bifurcation point $(0, \lambda_0)$,

$$F(x,\lambda) \approx x\left(\tfrac{1}{2}F_{xx}(0,\lambda_0)x - F_{x\lambda}(0,\lambda_0)\lambda\right)$$

alternates signs in a predictable way. Since S is bounded and doesn't approach any other trivial solutions, we can separate it from other zeros of F, including the trivial solutions, using the dashed curve (not really a curve!). But then, moving along this curve, we can connect a region where $F > 0$ to a region where $F < 0$ without ever passing through a point where $F = 0$, which is a contradiction.

3.2. **Generalization to higher dimension.** These ideas also generalize to higher dimensions. When $x \in \mathbb{R}^2$ you can use winding numbers, for $x \in \mathbb{R}^n$ you can use the Brouwer degree, and for x in a Banach space you can use the Leray–Schauder degree.

This last generalization was applied to our water wave problem as part of the proof of the famous "Stokes conjecture". At the ends of the bifurcation curves, one finds "extreme" waves with a sharp corner at each crest with a 120° interior angle.



References

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[2] J. F. Toland. Stokes waves. *Topol. Methods Nonlinear Anal.*, 7(1):1–48, 1996.