DAY 01

Administrative stuff

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office hours by appointment

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Class is M 1^{15} - 2^{45} , 3 - 3^{45}

15 min break

we can agree to shift things around, but for now seems fine

I can talk fast

class = long

- Always feel free to interrupt me or ask questions!
- No lecture notes, scribe system?
- *Prerequisites: complex analysis, ODEs, (multivariable calc)
- Non-prerequisites: PDEs, fun. anal, courses in physics/fluids
- Happy to spend some class time on background stuff, also in office hours / over email

Assesment: Last few weeks will be in-class presentations. Depending on time-constraints, 30-60 min.

Preliminary list of topics on the website. Other topics fine too. EMAIL ME AND WE WILL MEET IN MY OFFICE TO GET YOU STARTED.

Plan for today

1. Advertisement for mathematical fluid mechanics as a discipline
2. What we're going to do in this class
3. Start our way towards a derivation of the incompressible Euler equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla p \\ \nabla\cdot u = 0 \end{cases}$$

More about the journey than the destination.

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1. Fluids are everywhere, important
   - tea in a mug
   - blood in veins
   - plumbing
   - rivers, dams
   - oceans, lakes
   - atmosphere, weather
   - interior of the sun
   - interior of the earth

- Not cutting-edge theoretical physics. Mathematicians, Engineers, ...
- A big an important subfield of (applied) math. Many famous & well-studied PDEs have roots in fluid problems:
  - wave equation
  - Nonlinear Schrödinger
  - Korteweg-de Vries

  ⋮

- Clay Math millennium problem (~~\$~~ 1 million \$ prize):

> Do the Navier-Stokes eqns (in 3D) have smooth solns for smooth initial data, or do some solutions break down?

Lots of mathematical work on fluids, but still many tough open problems

2. This class

- Topics course in analysis, so we will be doing proofs, with ε's and δ's. We will do some functional analysis too, e.g. for

  Leray-Hopf weak solns

- ~~But~~ Sometimes we will just do "derivations" without a lot of rigor and sometimes w/ physical arguments, e.g.,

  derivation of Euler eqns today

- Sometimes we will look at explicit examples and do long calculations, e.g.

  ~~exam~~ complex analysis & potential flows

Fixed topics

- Derivation of E, N-S
- Potential flow in 2D
- Leray-Hopf weak solns in 3D

Likely topics

- strong solns in 2D
- Boundary layers
- Model eqns for water waves (NLS, KdV, ~~...~~ wave eqn)
- Steady 2D water waves

But I'm willing to shift focus based on ~~where~~ interests of the class.

I. Towards a derivation of the (incompressible) Euler equations

Models of fluids

1. Microscopic models

$$m_1 \ddot{x}_1 = f_1\begin{pmatrix} x_1, \ldots, x_N \\ \dot{x}_1, \ldots, \dot{x}_N, t \end{pmatrix}$$

$$m_2 \ddot{x}_2 = f_2(\ldots)$$

$$\vdots$$

$$m_N \ddot{x}_{3N} = f_N(\ldots)$$

m_i: mass of ith particle (kg)

$x_i(t)$: position (meters) $\mathbb{R}^n$

$\dot{x}_i(t)$: velocity (m/s)

$\ddot{x}_i(t)$: acceleration (m/s^2)

f_i: ~~forces~~ net force $\left(\frac{kg \cdot m}{s^2} = N\right)$

forces — external
- gravity

forces — ~~interact~~ b/w particles
- gravity
- electromagnetic

⋮

$\boxed{\text{Problem: } N \gg 1, \text{ say } N \sim 10^{23}}$

2. Kinetic description

Forget about the individual particles, and work probability density for a single particle (no correlation)

$$f(x, v, t)$$

position, velocity

$$f: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

$$\int_A \int_B f(x,v,t)\,dv\,dx$$

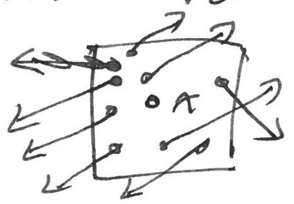
= "expected" # of particles which at time t have position $x \in A$ and velocity $\dot{x} = v \in B$.

Describing everything in terms of f is an approximation that only makes sense if we are interested in length scales longer than the average separation between particles.

Used for: stars in a galaxy, electrons in a plasma,

⋮

Note that particles at the "same location" x can have different velocities v.



Using f we can define hydrodynamical variables

$$\rho(x,t) = m \int f(x,v,t)\,dv$$

particle mass

= ~~average mass~~ mass density at x,

$$u(x,t) = \frac{\int v f(x,v,t)\,dv}{\int f(x,v,t)\,dv}$$

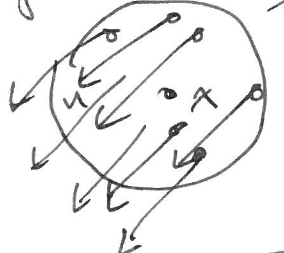
= average velocity at x,

⋮

3. Hydrodynamic models

Forget about f, just work with ~~the~~ hydrodynamic variables $\rho(x,t)$, $u(x,t)$, ...

Makes sense when f is nearly localized in v, i.e. particles have similar velocities.



This happens when "collisions" b/w particles are frequent.

$\boxed{\text{This continuum hypothesis is extremely accurate for a huge range of applications}}$

Eulerian & Lagrangian points of view

Suppose we are given a velocity field $u(x,t)$.
~~define~~ For simplicity take $x \in \mathbb{R}^n$
so $u : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$

A particle trajectory is a soln to the ODE

$$\begin{cases} \dot{X}(t) = u(X(t), t) \\ X(0) = X_0 \end{cases}$$

We write

$$\boxed{X(t) = \Phi^t(X_0)}$$

flow map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$

Basic calculations (chain rule)

$$\frac{d}{dt} g(X(t), t)$$

$$= \left(\sum_j \frac{\partial g}{\partial x_j} \frac{dX_j}{dt} + \frac{\partial g}{\partial t} \right)(X(t), t)$$

$$= \left(\sum_j \frac{\partial g}{\partial x_j} u_j + \frac{\partial g}{\partial t} \right)(\text{---})$$

$$= \frac{\partial g}{\partial t} + \underbrace{(u \cdot \nabla)}_{= \sum_j u_j \frac{\partial}{\partial x_j}} g \qquad (1)$$

Notation

'Lie derivative'

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + u \cdot \nabla$$

Called the material derivative (or convective derivative). It is a derivative following particle trajectories.

Summation convention

Repeated indices are summed over. E.g.

$u_i \frac{\partial}{\partial x_i}$ means $\sum_{i=1}^n u_i \frac{\partial}{\partial x_i}$

Matrix multiplication

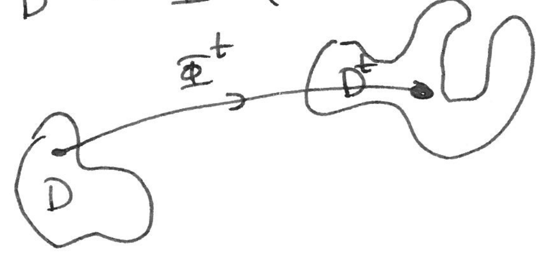
$$(Ax)_i = A_{ij} x_j$$

$$(AB)_{ij} = A_{ik} B_{kj}$$

Integral form of (1)

Instead of following a single particle, let's follow a whole set $D \subseteq \mathbb{R}^n$.

$$D^t := \Phi^t(D)$$



What is

$$\frac{d}{dt} \int_{D^t} g(x,t)\, dx \ ?$$

Suppose that t is small. Then $\Phi^0 = \mathrm{id}$ means that the Jacobian determinant

$$\det(D\Phi^t) > 0$$

Notation For $f: \mathbb{R}^n_x \times \mathbb{R}_t \to \mathbb{R}^m$,

$$(Df)_{ij} = \frac{\partial}{\partial x_i} f_j .$$

Usually $\nabla f = (Df)^T$.

Then we can change variables to get

$$\int_{D^t} g(x,t)\,dx = \int_{D^0} g(\Phi^t(x),t)\,\det(D\Phi^t(x))\,dx$$

$$D^0 \xrightarrow{\Phi^t} D^t$$

So

$$\frac{d}{dt}\int_{D^t} g(x,t)\,dx = \frac{d}{dt}\int_{D_0} g(\Phi^t,t)\,\det(D\Phi^t)\,dx$$

$$= \int_{D_0} \overbrace{\left(\frac{\partial g}{\partial t} + Dg\,\overset{u}{\boxed{\frac{\partial \Phi^t}{\partial t}}}\right)(\Phi^t,t)}^{\frac{Dg}{Dt}} \det(D\Phi^t)\,dx$$

$$+ \int_{D_0} g(\Phi^t,t)\,\boxed{\frac{\partial}{\partial t}\det(D\Phi^t)}\,dx$$

Aside: Derivatives of determinants

$$\det(I+\varepsilon B) = \det\begin{pmatrix} 1+\varepsilon b_{11} & & O(\varepsilon) \\ & \ddots & \\ O(\varepsilon) & & 1+\varepsilon b_{nn}\end{pmatrix}$$

$$= (1+\varepsilon b_{11})\det\begin{pmatrix} 1+\varepsilon b_{22} & O(\varepsilon) \\ O(\varepsilon) & \ddots\ 1+\varepsilon b_{nn}\end{pmatrix} + O(\varepsilon^2)$$

$$= \cdots = \prod_i (1+\varepsilon b_{ii}) + O(\varepsilon^2)$$

$$= 1 + \sum_i \varepsilon b_{ii} + O(\varepsilon^2)$$

So

$$\det(I+\varepsilon B) = \varepsilon \operatorname{tr} B + O(\varepsilon^2)$$

Similarly

$$\det(A+\varepsilon B) = \det(A(I+\varepsilon A^{-1}B))$$

$$= \det A \det(I+\varepsilon A^{-1}B)$$

$$= \det A\,(1+\varepsilon \operatorname{tr} A^{-1}B) + O(\varepsilon^2).$$

Thus

$$\frac{d}{dt}\det(A(t)) = \det A \det(A^{-1}A')$$

So!

$$\frac{\partial}{\partial t}\det(D\Phi^t(x)) = \det(D\Phi^t)\,\operatorname{tr}\left((D\Phi^t)^{-1}\frac{\partial D\Phi^t}{\partial t}\right)$$

Now

$$\frac{\partial \Phi^t}{\partial t} = u(\Phi^t,t)$$

$$\Rightarrow D\frac{\partial \Phi^t}{\partial t} = Du(\Phi^t,t)\,D\Phi^t$$

So

$$\frac{\partial}{\partial t}\det(D\Phi^t(x)) = \det(D\Phi^t)\,\operatorname{tr}\left(\cancel{(D\Phi^t)^{-1}}\,Du\,\cancel{D\Phi^t}\right)$$

$$= \det(D\Phi^t)\,\underbrace{\operatorname{tr}(Du)}_{\nabla\cdot u}$$

$=$

So we have

$$\frac{d}{dt}\int_{D^t} g(x,t)\,dx = \int_{D^t} \underline{\qquad}\,.$$

$$= \int_{D_0}\left(\frac{Dg}{Dt} + g(\nabla\cdot u)\right)(\Phi^t,t)\det(D\Phi^t)\,dx$$

~~Plugging in t=0 we conclude~~

$$\boxed{\frac{d}{dt}\int_{D^t} g(x,t)\,dx = \int_{D^t}\left(\frac{Dg}{Dt} + g(\nabla\cdot u)\right)dx}$$

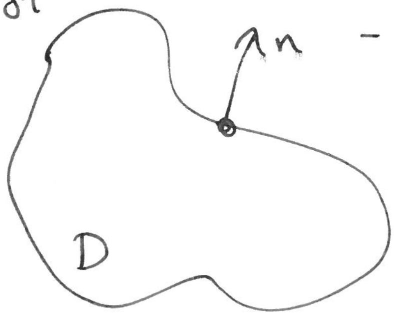
Conservation of mass

Suppose in addition to $u(x,t)$ we have a density field $\rho(x,t)$.

Eulerian derivation

$$\underbrace{\frac{d}{dt}\int_D \rho\,dx}_{\text{mass in } D} = \underbrace{-\int_{\partial D}\rho\,\underbrace{(u\cdot n)}_{\text{volume flux}}\,dS}_{\text{density flux}}$$

$$\int_D \frac{\partial\rho}{\partial t}\,dx \qquad = \qquad -\int_D \nabla\cdot(\rho u)\,dx$$



so

$$\int_D\left(\frac{\partial\rho}{\partial t} + \cancel{\rho(\nabla\cdot u)}\ \nabla\cdot(\rho u)\right)dx = 0$$

for all D

$$\therefore\ \boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho u) = 0}$$

Lagrangian derivation

Since mass is neither created nor destroyed, the mass of $D^t = \Phi^t(D)$ should be constant, i.e.

$$0 = \frac{d}{dt}\int_{D^t}\rho\,dx$$

$$= \int_{D^t}\left(\frac{D\rho}{Dt} + \rho(\nabla\cdot u)\right)dt.$$

So

$$0 = \frac{D\rho}{Dt} + \rho(\nabla\cdot u)$$

$$= \frac{\partial\rho}{\partial t} + (u\cdot\nabla)\rho + \rho(\nabla\cdot u)$$

$$= \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho u) \quad ✓$$

Incompressibility

In many applications, the density ρ is nearly constant along particle trajectories,

$$\frac{D\rho}{Dt} = 0$$

Equivalently, volume is preserved by the flow:

$$0 = \frac{d}{dt}\int_{D^t} 1\,dx = \int_{D^t}\left(\overset{\to 0}{\frac{D1}{Dt}} + 1(\nabla\cdot u)\right)dx$$

$$\Rightarrow \nabla\cdot u = 0.$$

These are equivalent since by mass conservation

$$\frac{D\rho}{Dt} = -\rho(\nabla\cdot u).$$