DAY 06

__Last time__ 2D Euler, finished existence & uniqueness for the Cauchy problem when $u_0 \in C^{1+\theta}$. Key fact used in proof was that

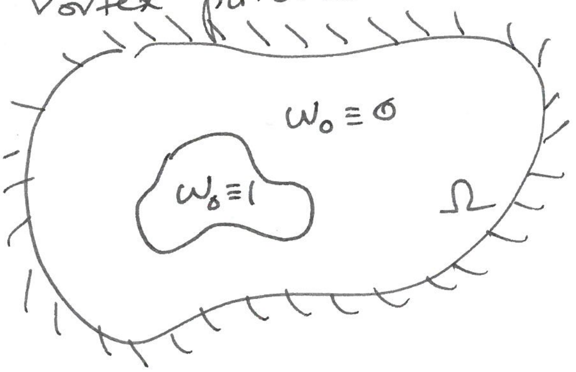
$$\frac{D\omega}{Dt} = 0.$$

Some remarks

- In 3D, one can check that
$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) u \not\equiv 0$$
and this argument completely breaks down.

- Can weaken the assumption $u_0 \in C^{1+\theta}$ to $\omega_0 \in L^\infty$, u_0 "quasi-Lipschitz". (Keep uniqueness.)

  __Ex__ Vortex patches

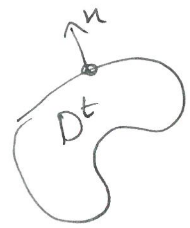


Today we leave the Euler eqns behind, at least for now, and talk about the Navier-Stokes equations.

__Recall__ We derived the momentum eqn in Euler by writing

$$\frac{d}{dt} \underbrace{\int_{D^t} \rho u \, dx}_{\text{momentum}} = \text{BODY FORCE} + \text{SURFACE FORCES}$$

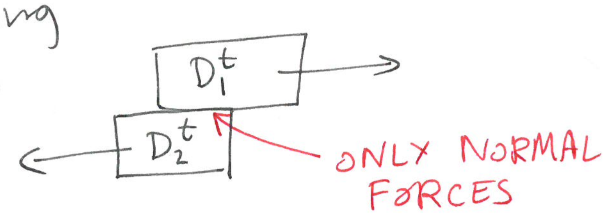
$$D^t = \Phi^t(D^0)$$



and __assuming__ that

$$\text{SURFACE FORCES} = -\int_{\partial D} p n \, dS$$

for some scalar function $p(x,t)$. This in particular means that there is no resistance to shearing



Now we will make a much more general ansatz:

$$\text{SURFACE FORCES} = \int_{\partial D} \Sigma(x, n, t) \, dS.$$

"stress" = force/area

$$\Sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

First, we will argue that Σ is linear in the normal vector n.

Recall that

$$\begin{matrix}\text{BODY} \\ \text{FORCES}\end{matrix} = \int_{D^t} f\,dx$$

for some force-density $f = f(x,t)$. Assuming that f, Σ are smooth, let's consider the scaled domains

$$D_\varepsilon := \varepsilon D.$$

space dimension

Then

$$\int_{D_\varepsilon} f\,dx = O(\varepsilon^{n})$$

$$\int_{\partial D_\varepsilon} \Sigma(x,n,t)\,dS = O(\varepsilon^{n-1})$$

$$\frac{d}{dt}\int_{D_\varepsilon} \rho u\,dx = O(\varepsilon^{n})$$

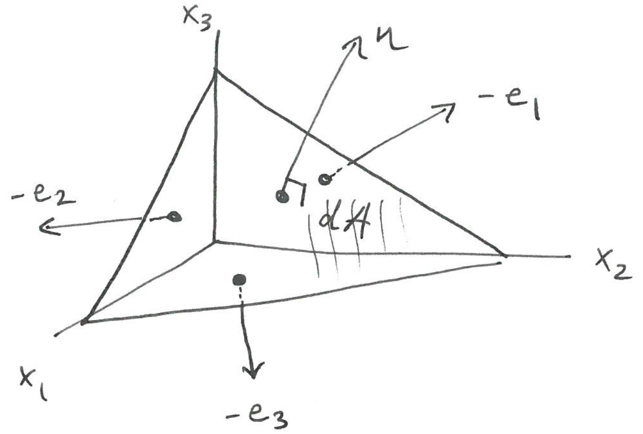
Since these all appear in the same eqn, deduce

$$\int_{\partial D_\varepsilon} \Sigma(x,n,t)\,dS = O(\varepsilon^{n}).$$

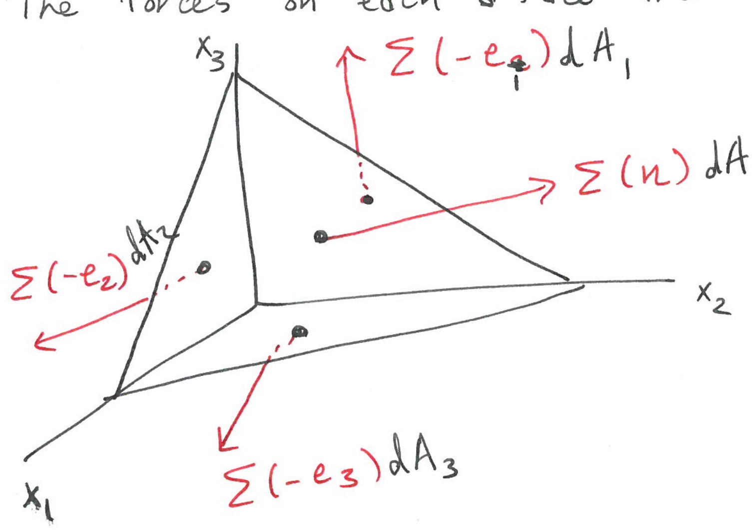
Now for a classic type of physics argument. There is surely a fancier way to do this, but I couldn't find a place where it's written down.

Consider an infinitesimal tetrahedron:

(so small, for instance, that Σ is essentially constant in x)



The forces on each face are



where the areas are

$$dA_i = n_i\,dA.$$

Our scaling argument shows that the net force must be zero, and so

$$\begin{aligned} 0 &= \Sigma(n)\,dA + \Sigma(-e_i)\,dA_i \\ &= \underbrace{\left[\Sigma(n) + \Sigma(-e_i)n_i\right]}_{=0} dA. \end{aligned}$$

Now write

$$\sigma_{ij} = -\Sigma_i(-e_i)\cdot e_j$$

to get

$$\Sigma(n)_i = \sigma_{ij} n_j$$

$$\left(\text{“}\Sigma = n\cdot\sigma\text{”}\right)$$

Since n was arbitrary, we conclude that

$$\boxed{\Sigma_i(x,n,t) = \sigma_{ij}(x,t)\, n_j(t)}$$

This argument also shows that σ_{ij} is a contravariant rank 2 tensor.

BRIEF INTERLUDE ON TENSORS

Say we have two coordinate systems x & $\bar{x}$. We can translate between them, viewing

$$x = x(\bar{x})$$

or alternatively

$$\bar{x} = \bar{x}(x).$$

Suppose we have a particle moving around,

$$x = X(t)$$
$$\bar{x} = \bar{X}(t)$$

with velocity

$$u = X'(t)$$
$$\bar{u} = \bar{X}'(t)$$

Then by the chain rule

$$\bar{u}_i = \frac{d}{dt}\bar{X}_i = \frac{d}{dt}\bar{x}_i(X(t)) = \frac{\partial \bar{x}_i}{\partial x_j}\frac{dX_j}{dt}$$

$$\boxed{\bar{u}_i = \frac{\partial \bar{x}_i}{\partial x_j} u_j}$$

This is the transformation rule for the components of a vector. (You have probably seen this in a class on manifolds.)

In general, a contravariant tensor

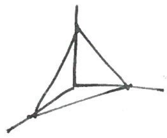
$$A_{i_1 i_2 \cdots i_k}$$

is an object with the transformation rule

$$\bar{A}_{i_1 i_2 \cdots i_k} = \frac{\partial \bar{x}_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial \bar{x}_{i_k}}{\partial x_{j_k}} A_{j_1 \cdots j_k}$$

So far we have assumed only that the net force due to Σ vanishes on an infinitesimal tetrahedron. Let's further assume that the net torque is zero:

$$0 = \int_{\partial T} x \times \Sigma(x,n,t)\, dS$$
$$= \int_{\partial T} e_i \epsilon_{ijk} x_j \sigma_{kl} n_l \, dS$$
$$= e_i \int_T \epsilon_{ijk} \frac{\partial (x_j \sigma_{kl})}{\partial x_l} dx$$



$$= e_i \int_T \epsilon_{ijk} \sigma_{kj}\, dx + \underbrace{e_i \int_T x_j \frac{\partial \sigma_{kl}}{\partial x_l} dx}_{\text{small b/c } |x| \text{ is small}}$$

We conclude that

$$\epsilon_{ijk} \sigma_{kj} = 0$$

$$\Longleftrightarrow \boxed{\sigma_{kj} = \sigma_{jk}}$$

In Euler we had

$$\sigma_{ij} = -p\,\delta_{ij}$$

and require $d_{ii} = 0$

In general, we write

$$\sigma_{ij} = -p\,\delta_{ij} + d_{ij}$$

We need to come up with a relationship b/w d_{ij} and $\frac{\partial u_k}{\partial x_l}$. These formulas can be pretty general (e.g. hypotheses for a Stokesian fluid), but we'll make a linear ansatz

$$d_{ij} = A_{ijkl} \frac{\partial u_k}{\partial x_l}.$$

Splitting

$$\frac{\partial u_k}{\partial x_l} = \underbrace{e_{kl}}_{\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)} - \frac{1}{2}\epsilon_{klm} \underbrace{\omega_m}_{(\nabla \times u)_m}$$

we get

$$d_{ij} = A_{ijkl} e_{kl} - \frac{1}{2} A_{ijkl} \epsilon_{klm} \omega_m$$

We now assume that this relationship is isotropic, i.e. that

$$\bar{A}_{ijkl} = A_{ijkl} \quad (*)$$

for any orthogonal change of variables with det = +1, really a rotation

"No preferred direction" (distinguished)

Fact $(*) \Rightarrow$

$$A_{ijkl} = \mu\,\delta_{ik}\delta_{jl} + \mu'\,\delta_{il}\delta_{jk} + \mu''\,\delta_{ij}\delta_{kl}. \quad (**)$$

Pf Easy to check that $(**)$ is isotropic. The reverse implication is a brute force calculation, really. □

In our case, since σ_{ij} is symmetric, A_{ijkl} should be symmetric in i & j, and so $\mu' = \mu$ in $(**)$. Then it is also symmetric in k & l, and we get

$$d_{ij} = A_{ijkl} e_{kl} - \cancel{\tfrac{1}{2} A_{ijkl} \epsilon_{klm} \omega_m} \; 0$$

$$= \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) e_{kl} + \mu''\,\delta_{ij}\delta_{kl} e_{kl}$$

$$= 2\mu e_{ij} + \mu''\,\delta_{ij} e_{kk}$$

$$(= 2\mu e_{ij} + \mu''\,\delta_{ij} (\nabla \cdot u))$$

Since we also wanted $d_{ii} = 0$, we get

$$0 = d_{ii} = 2\mu e_{ii} + \mu''\,\delta_{ii} e_{kk}$$
$$= 2\mu e_{ii} + 3\mu'' e_{ii}$$

$$\therefore\ 2\mu + 3\mu'' = 0$$

and so finally

"viscosity"

$$d_{ij} = 2\mu\left(e_{ij} - \frac{1}{3}\delta_{ij} \frac{\partial u_k}{\partial x_k}\right)$$

and hence, putting everything together

$$\boxed{\sigma_{ij} = -p\,\delta_{ij} + 2\mu\left(e_{ij} - \tfrac{1}{3}\delta_{ij} e_{kk}\right)}$$

~~Thus~~ Thus

$$\text{BODY FORCES} = \int_{\partial D} \Sigma_i(x,n,t)\, dS$$

$$= \int_{\partial D} e_i \sigma_{ij} n_j \, dS$$

$$= \int_D e_i \frac{\partial \sigma_{ij}}{\partial x_j}\, dx$$

$$= e_i \int_D \frac{\partial}{\partial x_j}\Big(-p\,\delta_{ij} + 2\mu\big(e_{ij} - \tfrac{1}{3}\delta_{ij} e_{kk}\big)\Big)\, dx$$

If the viscosity μ is constant, then the integrand is

$$\underbrace{-\frac{\partial p}{\partial x_j}\delta_{ij}}_{=-\frac{\partial p}{\partial x_i}} + 2\mu \underbrace{\frac{\partial e_{ij}}{\partial x_j}}_{=\mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} - \frac{2\mu}{3}\underbrace{\delta_{ij}\frac{\partial e_{kk}}{\partial x_j}}_{=\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)}$$

$$= -\frac{\partial p}{\partial x_i}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$= \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$= \mu\left(\Delta u_i + \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}\right)$$

$$= -\frac{\partial p}{\partial x_i} + \mu\left(\Delta u_i + \frac{1}{3}\frac{\partial}{\partial x_i}(\nabla\cdot u)\right)$$

For an incompressible fluid $\nabla\cdot u = 0$ and we are left with

$$\text{~~BODY~~ SURFACE FORCES} = -\nabla p + \mu \Delta u.$$

Thus our momentum eqn

$$\frac{d}{dt}\int_{D^t} \rho u \, dx = \text{BODY FORCES} + \text{SURFACE FORCES}$$

gives

$$\rho \frac{Du}{Dt} = f - \nabla p + \boxed{\mu \Delta u}$$

body force density

the new viscosity term

In particular, the incompressible Navier-Stokes eqns are

$$\boxed{(NS)\begin{cases}\rho \dfrac{Du}{Dt} = f - \nabla p + \mu \Delta u \\ \nabla\cdot u = 0\end{cases}}$$

Because (NS) has more derivatives of u than (E), it needs more boundary conditions. The most common choice is

$u =$ some prescribed function on $\partial\Omega$

and especially the no-slip condition

$$\boxed{u = 0 \quad \text{on } \partial\Omega}$$

This is based on the idea that fluid particles "adhere" to solid boundaries --- think of trying to clean a jar of honey or a bottle of olive oil.

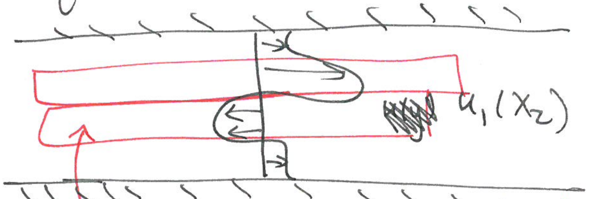
11 How important these new terms are, and the singular limit $\mu \to 0$, are topics for later. For now, let's do some examples!

2D shear flows ($\rho \equiv$ constant)

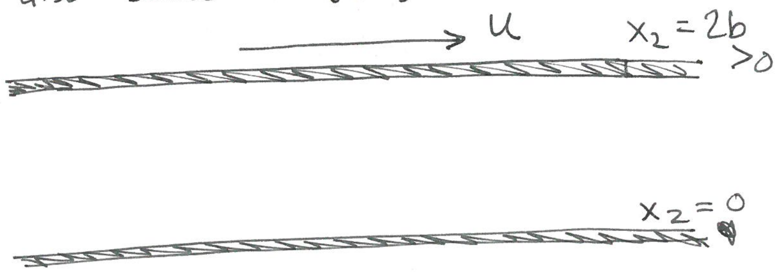
Recall that any flow (u_1, u_2) with

$$\begin{cases} u_1 = u_1(x_2) \\ u_2 = 0 \end{cases}$$

was a solution to the incompressible Euler equations



When are these shear flows also solns to (NS)?



Say the bottom plate is at rest, but the top plate is moving to the right with speed U. Then our BC are

$$\begin{cases} u_1 = U & \text{on } y = 2b \\ u_1 = 0 & \text{on } y = 0. \end{cases}$$

As with Euler, the incomp. condition $\nabla \cdot u = 0$ is automatically satisfied.

The momentum eqns are

$$\rho \cancel{\frac{\partial u_1}{\partial t}}^{0} + \rho \cancel{(u_1 \partial_1 + u_2 \partial_2) u_1}^{0} = -\partial_1 p + \mu \Delta u_1$$

$$\cancel{\rho \frac{\partial u_2}{\partial t}}^{0} + \cancel{\rho (u_1 \partial_1 + u_2 \partial_2) u_2}^{0} = -\partial_2 p + \cancel{\mu \Delta u_2}^{0}$$

i.e.

$$\begin{cases} \mu \dfrac{\partial^2 u_1}{\partial x_2^2} = \dfrac{\partial p}{\partial x_1} \\ \dfrac{\partial p}{\partial x_2} = 0 \implies p = p(x_1) \end{cases}$$

Integrating twice,

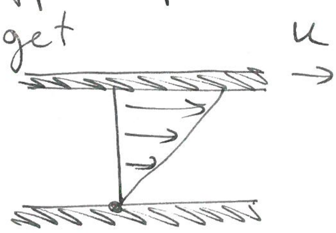
$$\mu u_1 = \frac{x_2^2}{2} \frac{dp}{dx_1} + C_1 + C_2 x_2$$

Plugging in the BC we get

$$\boxed{u_1 = \frac{x_2 U}{2b} + \frac{x_2}{\mu} \frac{dp}{dx} \left(b - \frac{x_2}{2} \right)}$$

Couette flow Suppose $p =$ const. Then we just get

$$u_1 = \frac{x_2 U}{2b}$$



Poiseuille flow Suppose $U = 0$ but $\frac{dp}{dx} = \text{const.} \neq 0$. Then

$$u_1 = -\frac{x_2}{\mu} \frac{dp}{dx} \left(b - \frac{x_2}{2} \right)$$