TYPO IN
DAY 01 NOTES

$(Df)_{ij}$ should be $\frac{\partial f_i}{\partial x_j}$

and not $\frac{\partial f_j}{\partial x_i}$

# DAY 02

**Recall** $x\in\mathbb{R}^n$, $t\in\mathbb{R}$

$\rho(x,t)$ = mass density
$u(x,t)$ = velocity field
$\Phi^t$ = flow map associated to $u$
$\Phi^t:\mathbb{R}^n\to\mathbb{R}^n$

$$\Phi^0 = id$$
$$\frac{\partial \Phi^t}{\partial t} = u(\Phi^t,t)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\cdot\nabla$$

$$\frac{d}{dt} g(\Phi^t(x),t) = \frac{Dg}{Dt}(\Phi^t(x),t)$$

If $D^t = \Phi^t(D^0)$, then

$$\frac{d}{dt}\int_{D^t} g(x,t)\,dx = \int_{D^t}\left[\frac{Dg}{Dt} + g(\nabla\cdot u)\right]dx$$

Conservation of mass:

$$0 = \frac{d}{dt}\int_{D^t}\rho\,dx = \int_{D^t}\left[\frac{D\rho}{Dt}+\rho(\nabla\cdot u)\right]dx$$

$$\therefore\ \boxed{\frac{D\rho}{Dt}+\rho(\nabla\cdot u)=0}$$

this is a more "Lagrangian" since it follows particle trajectories

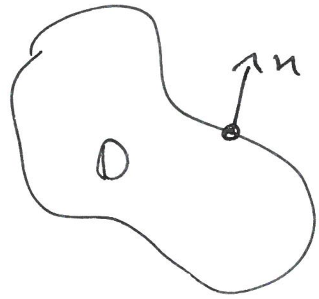


~~Alternate interpretation of incompressibility~~

## A more Eulerian derivation of mass conservation

$$\frac{d}{dt}\underbrace{\int_D \rho\,dx}_{\text{mass in } D} = -\int_{\partial D}\underbrace{\rho\overbrace{(u\cdot n)}^{\text{volume flux}}}_{\text{mass flux}}\,dS = -\int_D \nabla\cdot(\rho u)\,dx$$

fixed domain




Since $D$ is arbitrary, conclude that

$$\boxed{\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho u)=0}$$

## Incompressibility

In many applications (e.g. water in the ocean), the mass density $\rho$ is constant along particle trajectories:

$$\boxed{\frac{D\rho}{Dt}=0}$$

plug into

Let's ~~expand $\frac{D\rho}{Dt}$ and use~~ conservation of mass:

(1) $\frac{\partial\rho}{\partial t}+(u\cdot\nabla)\rho = 0$

(2) $\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho u)=0$

$$= \frac{\partial\rho}{\partial t}+\rho(\nabla\cdot u)+(u\cdot\nabla)\rho$$

$=0$

Assuming $\rho > 0$,

$$\boxed{\frac{D\rho}{Dt} = 0 \iff \nabla\cdot u = 0}$$

## Volume conservation

With $D^t = \Phi^t(D^0)$, have

$$\frac{d}{dt}\int_{D^t} 1\,dx = \int_{D^t}\left(\frac{D1}{Dt} + 1\cdot(\nabla\cdot u)\right)dx$$

$$= \int_{D^t}(\nabla\cdot u)\,dx.$$

~~Since $D^t$ is arbitrary,~~

So $0 = \frac{d}{dt}\int_{D^t} 1\,dx$ for all $D^t$ is equivalent to

$$\nabla\cdot u = 0.$$

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~~Motion of a point~~

## Conservation of momentum in the incompressible Euler equations

Remember that for the motion of a particle

$$m\ddot{x} = \frac{d}{dt}(\underbrace{m\dot{x}}_{\text{momentum}}) = F = \text{net force}$$

For us, the analogue of this momentum is

$$\int_{D^t} \rho u\,dx$$

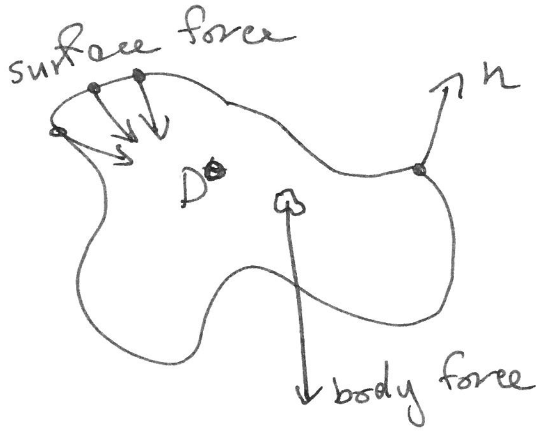
and so we expect

$$\frac{d}{dt}\int_{D^t} \rho u\,dx = \text{net force on } D^t$$



There are two types of forces to consider:

- body forces
- surface forces




**Assumption** There is a scalar pressure $p(x,t)$ so that

$$\text{surface forces} = -\int_{\partial D} p n\,dS$$

Units of $p$? Force/area.

[This is an excellent approximation in many cases.]

We write the body forces as

$$\text{body forces} = \int_D f\,dx$$

for some force density $f(x,t)$.

So the force/momentum balance is

$$\frac{d}{dt}\int_{D^t} \rho u\,dx = \int_{D^t} f\,dx - \int_{\partial D^t} p n\,dx$$

$$\int_{\partial D^t} p n\,dx = \int_{D} \nabla p\,dx$$

$$\frac{d}{dt}\int_{D^t} \rho u\,dx = \int_{D^t}\left(\frac{D(\rho u)}{Dt} + \rho u\,\cancel{(\nabla\cdot u)}\right)$$

$$= \int_{D^t}\left(\cancel{\frac{D\rho}{Dt}}\,u + \rho\frac{Du}{Dt}\right)dx$$

$$= \int_{D^t} \rho\frac{Du}{Dt}\,dx$$

Since $D^t$ is arbitrary,

$$\boxed{\rho\frac{Du}{Dt} = f - \nabla p}$$

The incompressible Euler equations:

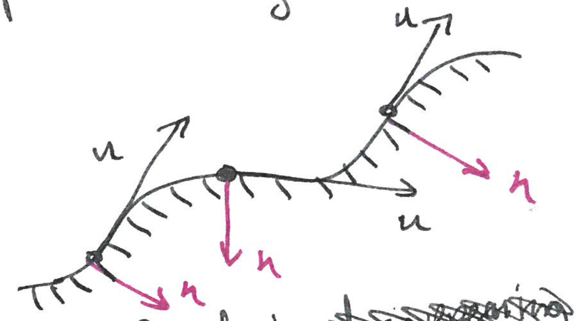
$$\begin{cases} \rho\dfrac{Du}{Dt} = -\nabla p + f & \text{(momentum)} \\ \nabla\cdot u = 0 & \text{(incompressible)} \\ \dfrac{D\rho}{Dt} = 0 & \text{(mass conservation)} \end{cases}$$



Typical boundary condition

$u\cdot n = 0$ at rigid walls

"Slip boundary condition"




otherwise fluid is ~~disappearing~~ being created / destroyed.

A class of examples:
Shear flows

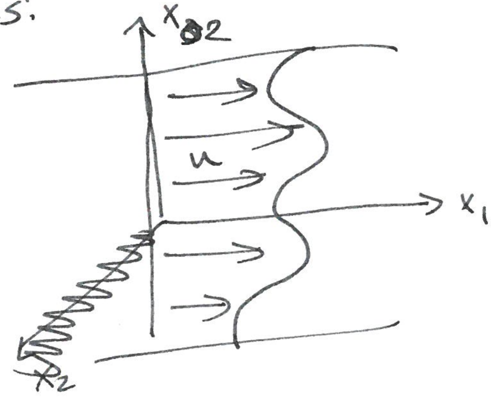
With no body forces $f$,
suppose

$$\begin{cases} u(x_1,x_2,x_3,t) = \begin{pmatrix} u_1(x_2,x_3) \\ 0 \\ 0 \end{pmatrix} \\ \rho = \rho(x_2,x_3) \\ p = 0 \end{cases}$$

Then

$$\begin{cases} \nabla\cdot u = \cdots = 0 \\ \dfrac{D\rho}{Dt} = \cdots = 0 \\ \rho\dfrac{Du}{Dt} = \cdots = 0 \\ \nabla p = 0 \end{cases}$$

and so this is a soln to the incompressible Euler equations.



# Motion near a point, Vorticity

So far we've been deriving PDEs for $u$ & $\rho$. I want to start looking at explicit solutions of these PDEs so that we can really get our hands on something.

But first we need to briefly talk about vorticity $\omega = \nabla \times u = \text{curl}\, u$. We will take a more in-depth look next week.

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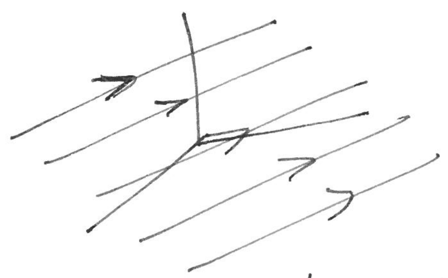
Near a fixed point, WLOG near $x = 0$, we have

$$u(x) = u(0) + Du(0)x + O(|x|^2)$$
$$= u(0) + \frac{Du(0) + Du(0)^T}{2} + \frac{Du(0) - Du(0)^T}{2} + O(|x|^2)$$

The first term just represents uniform motion



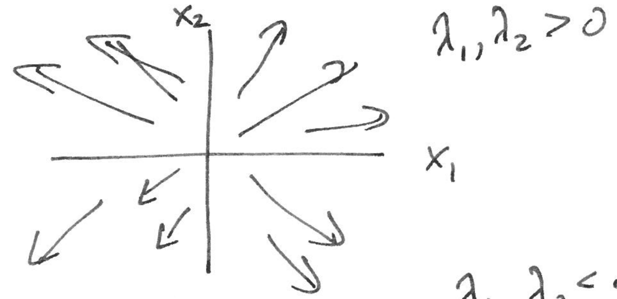
but the next two are more interesting.

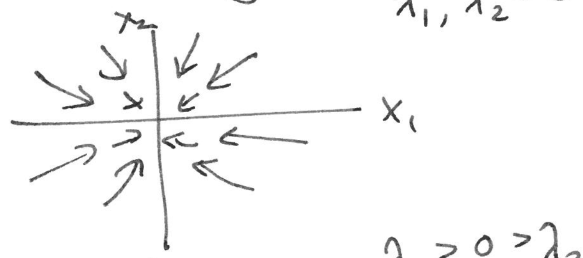


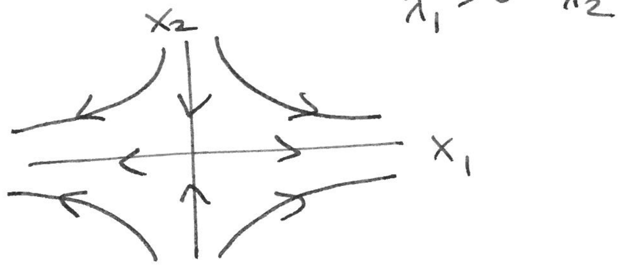
$\frac{Du + (Du)^T}{2}$ is symmetric, so up to a change of coordinates it is

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Now $\lambda_1 + \lambda_2 + \lambda_3 = \nabla \cdot u(0)$ is zero for incompressible flows, but otherwise there is no restriction.




This is called straining motion, and $\frac{Du + Du^T}{2}$ is called the rate of strain tensor.

Let's look at the antisymmetric term

$$\frac{Du - (Du)^T}{2}$$

**More index notation**

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Kronecker $\delta$-fun

$$\epsilon_{ijk} = \begin{cases} +1 & (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & (i,j,k) \text{ is an odd perm. of } (1,2,3) \\ 0 & \text{else} \end{cases}$$

Levi-Civita symbol

even/odd means an even/odd # of swaps:

1. $(1,2,3) \to (1,2,3)$. No swaps, so even.
2. $(1,2,3) \to (2,1,3)$ 1 swap, so odd.
3. $(1,2,3) \to (2,3,1)$ 2 swaps, even

$$\boxed{\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}}$$

$$(A \times B)_i = \epsilon_{ijk} A_j B_k$$

$$(\nabla \times A)_i = \epsilon_{ijk} \partial_j A_k$$



**Claim**

$$\frac{Du - (Du)^T}{2} x = \frac{1}{2}\omega \times x$$

where $\omega = \nabla \times u =$ vorticity

**Pf**

$$\left(\frac{Du - (Du)^T}{2}\right)_{ij} = \frac{-\partial_i u_j + \partial_j u_i}{2}$$

$$\left(\frac{Du - (Du)^T}{2} x\right)_i = \boxed{\frac{-\partial_i u_j + \partial_j u_i}{2} x_j}$$

$$\omega_i = \epsilon_{ijk}\partial_j u_k$$

$$\left(\tfrac{1}{2}\omega \times x\right)_i = \tfrac{1}{2}\epsilon_{ijk}\omega_j x_k$$

$$= \tfrac{1}{2}\epsilon_{ijk}\epsilon_{jlm}\partial_l u_m x_k$$

$$= -\tfrac{1}{2}\epsilon_{jik}\epsilon_{jlm}\partial_l u_m x_k$$

$$= -\tfrac{1}{2}(\delta_{il}\delta_{km} - \delta_{im}\delta_{lk})(\partial_l u_m) x_k$$

$$= \boxed{-\tfrac{1}{2}(\partial_i u_k) x_k + \tfrac{1}{2}(\partial_k u_i) x_k}. \quad \square$$

What do solns to

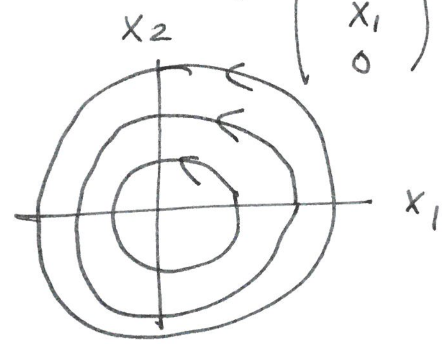
$$\dot{x} = \frac{1}{2}\omega \times x$$

$\omega \in \mathbb{R}^3$, fixed

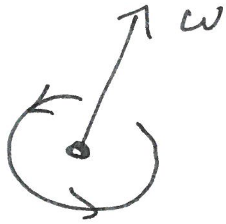
look like?

Well after a rotation & scaling, $\omega = (0, 0, 2)$ and so

$$\frac{1}{2}\omega \times x = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 0 & 1 \\ x_1 & x_2 & x_3 \end{vmatrix} = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$



In general, $\dot{x} = \frac{1}{2}\omega \times x$ is rotation about the vector $\frac{\omega}{|\omega|}$ with angular velocity $|\omega|$



Summing up, we can decompose fluid motion near a point as

$$u(0) + \frac{Du + Du^T}{2} + \frac{Du - Du^T}{2} + \dots$$

= translation + strain + rotation

<u>Definition</u> We call $u$ <u>irrotational</u> if $\omega = \nabla \times u \equiv 0$.

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$\omega$ has many interesting properties, but for now let's focus on irrotational flow

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<u>Irrotational and/or Incompressible motion: velocity potentials and stream functions</u>

$\boxed{3D}$ Suppose $\nabla \times u = 0$. Then locally $u = \nabla\phi$ where $\phi$ ← velocity potential

If $\nabla \cdot u = 0$, then $\Delta\phi = 0$

$\boxed{2D}$ Switch notation

$$\vec{u} = (u, v)$$
$$\vec{x} = (x, y)$$

Then $\nabla \cdot \vec{u} = 0 \iff$

$$\text{curl}(-v, u) = u_x + v_y = 0$$
$$\iff (-v, u) = \nabla\psi \ \text{(locally)}$$

stream function

Suppose $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} \equiv 0$. Then

$$\frac{D\psi}{Dt} = (u, v) \cdot \nabla\psi = (u, v) \cdot (-v, u) = 0.$$

$\boxed{\psi \text{ is constant along particle trajectories in steady flow}}$

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<u>Steady Potential flow in 2D, complex variables</u>

Suppose $u = u(x, y)$, $v = v(x, y)$ describe an irrotational and incompressible flow.

Then

$$\left.\begin{aligned} u_x + v_y &= 0 \\ v_x - u_y &= 0 \end{aligned}\right\} \iff u - iv \text{ holomorphic fun. of } z = x + iy$$

Similarly

$$\left.\begin{aligned} \phi_x - \psi_y &= 0 \\ \psi_x + \phi_y &= 0 \end{aligned}\right\} \iff \phi + i\psi \text{ holo.}$$

Moreover

$$\frac{d(\phi + i\psi)}{dz} = u - iv$$