DAY 07

<u>Last time</u> we derived the (constant density, incompressible) Navier–Stokes eqns

$$(NS)\quad \begin{cases} \partial_t u + (u\cdot\nabla)u - \underbrace{\nu}_{\text{const.}>0}\Delta u + \nabla p = f \\ \nabla\cdot u = 0 \\ u|_{t=0} = u_0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

This is like the Euler eqns we studied before, but with an extra $-\nu\Delta u$ term in the momentum eqn. Now the momentum eqn almost looks like the heat eqn:

$$\partial_t u - \nu\Delta u = \text{other terms}$$

Thus we expect some diffusion of u, maybe even some smoothing, contrast to the Euler eqns where we ~~only~~ have transport.

<u>Today and next time</u> we will study <u>weak solns</u> to the Cauchy problem (NS), following Chemin's book cited on the website (can download PDF through the library).

This is very different from our look at 2D Euler

- Solutions are nonunique
- Sobolev spaces & Hilbert space techniques

The fundamental bound for smooth solns of (NS) is the <u>energy equality</u> which we get by testing the equations against u:

$$\int_\Omega u\cdot\partial_t u\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx$$

$$\begin{aligned}\int_\Omega u\cdot[(u\cdot\nabla)u]\,dx &= \boxed{\int_\Omega u_i u_j \partial_j u_i\,dx} \\ &= -\int_\Omega \partial_j(u_i u_j)u_i\,dx + \underbrace{0}_{u|_{\partial\Omega}\equiv 0} \\ &= -\int_\Omega \cancelto{0}{(\partial_j u_j)} u_i u_i\,dx \\ &\quad - \boxed{\int_\Omega u_j(\partial_j u_i)u_i\,dx}\end{aligned}$$

$$\therefore \int_\Omega u\cdot[(u\cdot\nabla)u]\,dx = 0$$

$$\begin{aligned}\int_\Omega u\cdot\Delta u\,dx &= \int_\Omega u_i\,\partial_j\partial_j u_i\,dx \\ &= -\int_\Omega (\partial_j u_i)(\partial_j u_i)\,dx \\ &= -\int_\Omega |\nabla u|^2 dx\end{aligned}$$

Finally,

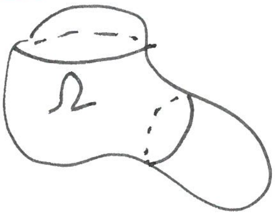
$$\begin{aligned}\int_\Omega u\cdot\nabla p\,dx &= \int_\Omega u_i\partial_i p\,dx \\ &= -\int_\Omega \cancelto{0}{(\partial_i u_i)}\,p\,dx = 0.\end{aligned}$$

Putting it all together and integrating w.r.t. t we get

$$\boxed{\begin{aligned}\frac{1}{2}\int_\Omega |u|^2 dx + \nu\int_0^t\int_\Omega |\nabla u|^2 dx\,dt' \\ = \frac{1}{2}\int_\Omega |u_0|^2 dx + \int_0^t\int_\Omega f\cdot u\,dx\,dt'\end{aligned}}$$

So if $f \equiv 0$, for instance, we expect $\int_\Omega |\nabla u|^2 dx \longrightarrow 0$ as $t \longrightarrow \infty$.

THE PROOF IN CHEMIN'S BOOK APPLIES TO QUITE GENERAL Ω. FOR SIMPLICITY, WE WILL $\boxed{\text{ALWAYS ASSUME}}$ Ω IS BDD.



$\Omega \subseteq \mathbb{R}^d$

$d = 2$ or 3

§1 Some elements of functional analysis

[CHEMIN IS CAREFUL HERE AND PROVIDES LOTS OF PROOFS. VERY READABLE.]

Definitions

$$|f|_{H^1}^2 := |f|_{L^2}^2 + |\nabla f|_{L^2}^2$$

$$\mathcal{D} := C_0^\infty(\Omega)$$

$\mathcal{D}' :=$ distributions

$$\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\mathcal{D}' \times \mathcal{D}}$$

$$H_0^1 := \overline{\mathcal{D}}^{H^1}$$

$H^{-1} := \mathcal{D}' \times \mathcal{D}$ dual of H_0^1

$$|f|_{H^{-1}} := \sup_{|\varphi|_{H^1} \le 1} \langle f, \varphi \rangle$$

Poincaré inequality $\exists C = C(\Omega)$ s.t. $|f|_{L^2} \le |\nabla f|_{L^2}$ $\forall f \in H_0^1$.

In particular, we can use $(f, g)_{H_0^1} := (\nabla f, \nabla g)_{L^2}$ as an inner product on H_0^1.

More definitions

$V := (H_0^1)^d$ (= H_0^1 vector fields)

$V_\sigma := \{ v \in V : \nabla \cdot v = 0 \}$, a closed subspace of V.

$V' := (H^{-1})^d$, $\mathcal{H} := \overline{V_\sigma}^{L^2}$

For $f \in V'$ and $v \in V$, $\langle f, v \rangle := \langle f_j, v_j \rangle$.

Polar spaces

For subspaces $E \subseteq V$, $F \subseteq V'$,

$$E^0 := \{ f \in V' : \langle f, v \rangle = 0 \ \forall v \in E \}$$

$$F^* := \{ v \in V : \langle f, v \rangle = 0 \ \forall f \in F \}$$

V' is a Hilbert space with $|f|_{V'} = \sup_{|v|_V \le 1} \langle f, v \rangle$.

Leray projector

$\mathbb{P} :=$ orthogonal projection of $(L^2)^d$ onto $\mathcal{H}$.

Stokes operator

We say $Au = f$ for $u \in V_\sigma$ and $f \in V'$ if

$$\langle -\Delta u, v \rangle = \langle f, v \rangle \quad \forall v \in V_\sigma,$$

i.e. if $-\Delta u - f \in V_\sigma^0$.

Think of $Au = f$ as a weak version of

$$\begin{cases} -\Delta u + \nabla p = f \\ \nabla \cdot u = 0 \end{cases}$$

Thm 1.1 $\forall f \in V' \ \exists!\, u \in V_\sigma$ s.t. $Au = f$.

Pf (Easy!) Let $f \in V'$. Then $v \mapsto \langle f, v\rangle$ is a linear functional on $V_\sigma \subseteq V$. Thus by Riez $\exists!\, u \in V_\sigma$ s.t.

$$(u,v)_{H^1_0} = \langle f, v\rangle \quad \forall v \in V_\sigma$$

$$\| \quad (\nabla u, \nabla v)_{L^2}$$

$$\| \quad \langle -\Delta u, v\rangle$$

$\square$

Prop 1.1 $\forall f \in V' \ \exists f_n \in \mathcal{H}$ s.t. $f_n \to f$ in $V_\sigma' :=$ bold linear functionals on V_σ

Pf $(L^2)^d \subseteq V' = (H^{-1})^d$ is dense, and so $\exists\, g_n \in (L^2)^d$ with $g_n \to f$ in V'. ~~Thus~~ In particular, since $V_\sigma \subseteq V$, $g_n \to f$ in V_σ'. Set $f_n = \mathbb{P} g_n \in \mathcal{H}$. Then

$$|f - f_n|_{V_\sigma'} \le |g_n - \mathbb{P} g_n|_{V_\sigma'} + \underbrace{|g_n - f|_{V'}}_{\to 0}$$

$$= \sup_{|v|_{V_\sigma} \le 1} (g_n - \mathbb{P} g_n, v)_{L^2}$$

$$= \sup_{|v|_{V_\sigma} \le 1} \big(g_n, \overbrace{(I - \underbrace{\mathbb{P}}_{\text{orthogonal proj}})v}^{0}\big)_{L^2}$$

$$= 0.$$

$\square$

Gagliardo–Nirenberg (Cor 1.2)

For $2 \le p < \infty$ with $\frac{1}{p} > \frac{1}{2} - \frac{1}{d}$, $\exists C = C(p,d)$ s.t.

$$|u|_{L^p} \le C\, |u|_{L^2}^{1-\sigma}\, |Du|_{L^2}^{\sigma},$$

$$\sigma := \frac{d(p-2)}{2p} \quad \text{(Proof in book)}$$

Thm 1.3 $H^1_0 \hookrightarrow L^2$ and $L^2 \hookrightarrow H^{-1}$ are compact. (Proof in book)

Prop 1.2 $\forall g \in V_\sigma^\circ \ \exists p \in L^2$ s.t. $g = \nabla p$.

(Proof in book)

§2 Existence of weak solns

§2.1 Spectral properties of the Stokes operator (A)

~~In Thm 1.1 we showed that~~ We want to understand the Stokes operator spectrum of the ~~A~~. As is often done with elliptic operators, we will actually look at its inverse:

~~Def For~~

$$\begin{tikzcd}
(L^2)^d \arrow[d, phantom, "\cup"] & & & (L^2)^d \arrow[d, phantom, "\cup"] \\
\mathcal{H} \arrow[r, hook] \arrow[rrr, bend right, "B"'] & V' \arrow[r, "A^{-1}"] & V_\sigma \arrow[r, hook] & \mathcal{H}
\end{tikzcd}$$

Prop 2.1 B is continuous, compact, self-adjoint, positive, one-to-one. Thus $\overline{\operatorname{ran} B} = (\ker B)^\perp = \mathcal{H}$.

Pf

Symmetry Let $f, g \in \mathcal{H}$, $u = \mathcal{B}f \in \mathcal{V}_\sigma$
$v = \mathcal{B}g \in \mathcal{V}_\sigma$

Then

$$(Bf, g)_{L^2} = \langle -\Delta v, u\rangle = (\nabla u, \nabla v)_{L^2} = \langle -\Delta u, v\rangle = (f, Bg)_{L^2}.$$

Bdd, pos, compact

$$|u|^2_{\mathcal{V}_\sigma} = |\nabla u|^2_{L^2} = \langle -\Delta u, u\rangle$$

$$|Bf|^2_{\mathcal{V}_\sigma} = \langle f, \mathcal{B}f\rangle = (f, Bf)_{L^2} \le |Bf|_{L^2}|f|_{L^2} \le C|Bf|_{\mathcal{V}_\sigma}|f|_{L^2}$$

Thus $|Bf|_{\mathcal{V}_\sigma} \le C|f|_{L^2}$, and B is bdd $\mathcal{H} \to \mathcal{V}_\sigma$, hence compact $\mathcal{H} \to \mathcal{H}$. Moreover, we have

$$(Bf, f)_{L^2} = |\nabla u|^2_{L^2} \ge c|u|^2_{L^2}$$

by Poincaré, and so $\mathcal{B}$ is positive.

One-to-one $f \in \ker B \iff -f \in \mathcal{V}_\sigma^0$

$\iff \langle f, v\rangle = 0 \;\; \forall v \in \mathcal{V}_\sigma$

$\langle f, v\rangle = (f, v)_{L^2} = (f, v)_{\mathcal{H}}$.

Thus $f \in (\mathcal{V}_\sigma)^\perp \subseteq \mathcal{H}$. But $\mathcal{H} = \overline{\mathcal{V}_\sigma}^{L^2}$, and so $(\mathcal{V}_\sigma)^\perp = \{0\}$.

Range

$$\overline{\operatorname{ran} B} = (\ker \mathcal{B}^*)^\perp = (\ker B)^\perp = \{0\}^\perp = \mathcal{H}.$$

□

Thm 2.1 $\exists$ an ONB $\{e_k\}$ of $\mathcal{H}$ and non-decreasing

$$0 < \mu_k^2 \to \infty$$

$\mu_k \sim$ freq. of e_k
$\Delta e^{i\lambda x} = -\lambda^2 e^{i\lambda x}$

s.t.

$$(*) \quad -\Delta e_k + \nabla \pi_k = \mu_k^2 e_k$$

for some $\pi_k \in L^2_{loc}$ with $\nabla \pi_k \in \mathcal{V}_\sigma^0$.

Moreover, $\{\mu_k^{-1} e_k\}$ is an ONB of $\mathcal{V}_\sigma$ with the H_0^1 inner product, and for $f \in V'$,

$$|f|^2_{V'_\sigma} = \sum \mu_j^{-2} \langle f, e_j\rangle^2.$$

Pf Since $\mathcal{B}: \mathcal{H} \to \mathcal{H}$ is positive and compact, (Prop 2.1) $\exists$ an ONB of $\mathcal{H}$, $\{e_k\}$, and $0 < \mu_k^{-2} \searrow 0$ s.t. $\mathcal{B}e_k = \mu_k^{-2} e_k$.

$(*)$ follows from Prop 1.2. Note that $e_k \in \mathcal{V}_\sigma$.

Next let's look at $\{\mu_k^{-1} e_k\} \subseteq \mathcal{V}_\sigma$. Calculate

$$(e_k, e_j)_V = (\nabla e_k, \nabla e_j)_{L^2} = \langle -\Delta e_k, e_j\rangle = \langle -\nabla \pi_k + \mu_k^2 e_k, e_j\rangle = \mu_k^2 (e_k, e_j)_{L^2} = \mu_k^2 \delta_{kj}.$$

So we just need to show that $\operatorname{span}\{e_k\} \subseteq \mathcal{V}_\sigma$ is dense. But

$$(e_k, u)_{\mathcal{V}_\sigma} = (\nabla e_k, \nabla u)_{L^2} = \langle -\Delta e_k, u\rangle = \mu_k^2 \langle e_k, u\rangle = \mu_k^2 (e_k, u)_{L^2}$$

and so $(e_k, u)_{\mathcal{V}_\sigma} = 0 \;\; \forall k$

$\Rightarrow (e_k, u)_{L^2} = 0 \;\; \forall k$

$\Rightarrow u = 0$.

Finally

$$|f|^2_{V'_\sigma} = \sup_{|v|_{V_\sigma}\le 1} \langle f,v\rangle^2$$

$$\text{(DUALITY)} = \sup_{\alpha\in\ell^2}\sup_k \Big\langle f, \sum_{j\le k}\alpha_j\mu_j^{-1}e_j\Big\rangle^2$$

$$= \sup_{\alpha\in\ell^2}\sup_k\Big(\sum_{j\le k}\alpha_j\mu_j^{-1}\langle f,e_j\rangle\Big)^2$$

$$= \big\|\mu_j^{-1}\langle f,e_j\rangle\big\|^2_{\ell^2_j}. \qquad \square$$

<u>Def</u> (Frequency cutoff)

$$\mathbb{P}_\lambda : V' \longrightarrow V_\sigma$$

$$\mathbb{P}_\lambda f = \sum'_{\{j:\mu_j^2<\lambda\}} \langle f,e_j\rangle e_j$$

"~~cuts~~ Projects away frequencies $\gtrsim\sqrt{\lambda}$"

Since $\{e_k\}$ is an ONB for H, $\forall v\in(L^2)^d$ we have

$$\boxed{\mathbb{P}v = \sum_j \langle v,e_j\rangle e_j = \lim_{\lambda\to\infty}\mathbb{P}_\lambda v}$$

§2.1 The Leray theorem

<u>Def</u> We call u a global weak soln of (NS) with forcing $f\in L^2_{loc}(\mathbb{R}^+,V')$ if

$$\left.\begin{aligned} u\in C(\mathbb{R}^+,V'_\sigma)&\cap L^\infty_{loc}(\mathbb{R}^+,H)\\ &\cap L^2_{loc}(\mathbb{R}^+,V_\sigma)\end{aligned}\right\}$$

tradeoff b/w t & x regularity

and $\forall\Psi\in C^1(\mathbb{R}^+,V_\sigma)$ we have

$$(S\Psi)\quad \int_\Omega u\cdot\Psi\,dx + \int_0^t\int_\Omega\Big(\nu\nabla u:\nabla\Psi - u\otimes u:\nabla\Psi - u\cdot\partial_t\Psi\Big)dx\,dt'$$

$$= \int_\Omega u_0\cdot\Psi\big|_{t=0}\,dx + \int_0^t\langle f,\Psi\rangle dt'$$

$$\left(\begin{aligned}\nabla u:\nabla\Psi &= (\partial_i u_j)(\partial_i\Psi_j)\\ u\otimes u:\nabla\Psi &= u_iu_j\partial_i\Psi_j\end{aligned}\right)$$

<u>Thm</u> (Leray-Hopf)

$\forall u_0\in H$ $\exists$ a global weak soln of (NS). Moreover, this soln satisfies the ENERGY INEQUALITY

$$\frac12\int|u|^2dx + \nu\int_0^t\int|\nabla u|^2dx\,dt' \le \frac12\int|u_0|^2dx + \int_0^t\langle f,u\rangle dt'$$

<u>Def</u> weak global solns satisfying the energy inequality are called <u>Leray solutions</u>.

<u>Prop 2.5</u> $\exists C=C(\Omega)$ s.t. Leray solns satisfy

$$|u|^2_{L^2}+\nu\int_0^t|\nabla u|^2_{L^2}dt' \le |u_0|^2_{L^2} + \frac{C}{\nu}\int_0^t|f|^2_{V'_\sigma}dt'$$

<u>Pf</u> Use energy inequality and

$$\langle f,u\rangle \le |f|_{V'_\sigma}|u|_V = |f|_{V'_\sigma}|\nabla u|_{L^2} \le \nu|\nabla u|^2_{L^2}+\frac{C}{\nu}|f|^2_{V'_\sigma}$$

$\square$