DAY 03

All day today, we will assume that $\rho \equiv \text{const}$.

Recall Incompressible Euler eqns:

$$\star \begin{cases} \underbrace{\frac{\partial u}{\partial t} + (u\cdot\nabla)u}_{\frac{Du}{Dt}} = -\frac{\nabla p}{\rho} \\ \nabla\cdot u = 0 \end{cases}$$

BC $u\cdot n = 0$ at rigid walls

outward normal

Vorticity $\omega = \nabla\times u$

Irrotational $\Leftrightarrow \omega \equiv 0$,

$\Rightarrow u = \nabla\phi$ (locally)

Steady 2D potential flow

In 2D, write

$\vec{u} = (u,v)$

$\vec{x} = (x,y)$

$z = x+iy$

Incompressibility + irrotationality imply that

$u - iv$ is holomorphic

Moreover, locally $\exists\ \phi, \psi$ s.t.

$\phi_x = \psi_y = u$,

$\phi_y = -\psi_x = v$,

velocity potential

stream function

i.e. $\frac{d(\phi+i\psi)}{dz} = u - iv$

For steady flow $\left(\frac{\partial}{\partial t} \equiv 0\right)$,

$$\frac{D\psi}{Dt} = 0$$

and so

$$\boxed{\text{Level curves of } \psi \longleftrightarrow \text{particle trajectories}}$$

Bernoulli's law Suppose $\rho \equiv \text{const}$ and $\underbrace{u = \nabla\phi}_{\text{irrot.}}$, $\underbrace{\nabla\cdot u = 0}_{\text{incomp.}}$

Then (u,p) solves the incomp. Euler eqns $\star$ where p is defined by

$$\boxed{\underbrace{\rho\frac{|\nabla\phi|^2}{2}}_{\text{kinetic energy density}} + p + \rho\phi_t = \text{constant}. \quad (\star\star)}$$

Pf Applying ∇ to $(\star\star)$ yields

$$\rho\, \nabla\frac{|\nabla\phi|^2}{2} + \nabla p + \boxed{\rho\nabla\phi_t} = 0$$

$$= \rho\frac{\partial}{\partial t}\nabla\phi = \rho\frac{\partial u}{\partial t}$$

$$\left(\nabla\frac{|\nabla\phi|^2}{2}\right)_i = \partial_i\left(\frac{\partial_j\phi\,\partial_j\phi}{2}\right)$$

$$= [\partial_i(\partial_j\phi)](\partial_j\phi)/2 + (\partial_j\phi)[\partial_i(\partial_j\phi)]/2$$

$$= (\partial_j\phi)(\partial_{ij}\phi)$$

$$= \underbrace{(\partial_j\phi\,\partial_j)}_{u\cdot\nabla}\underbrace{(\partial_i\phi)}_{u_i}$$

$$= [(u\cdot\nabla)u]_i. \qquad \square$$

Application to 2D steady pot. flow

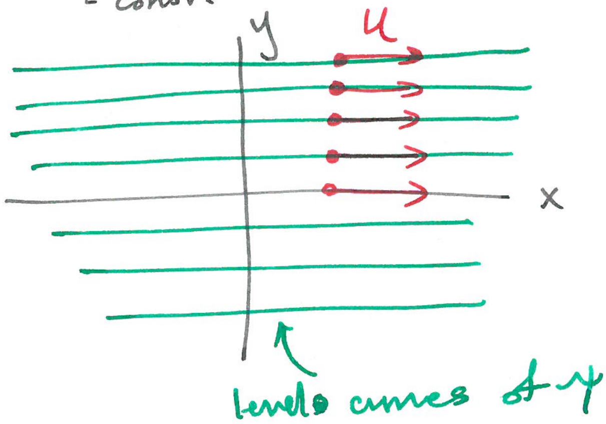
Any holomorphic $\phi + i\psi$ or $u - iv$ corresponds to a steady soln of the incomp. Euler equations (!)

Example 1 $\phi + i\psi = Uz,\ U \in \mathbb{R}$

$$u - iv = \frac{d}{dz} Uz = U \in \mathbb{R}$$

$$\psi = \mathrm{Im}(Uz) = Uy$$

$$p = \rho \frac{|u-iv|^2}{2} + \text{const.}$$
$$= \rho \frac{U^2}{2} + \text{const.}$$
$$= \text{const.}$$



Example 2 $\phi + i\psi = U\left(z + \frac{a^2}{z}\right)$ for $|z| > a \in \mathbb{R}$

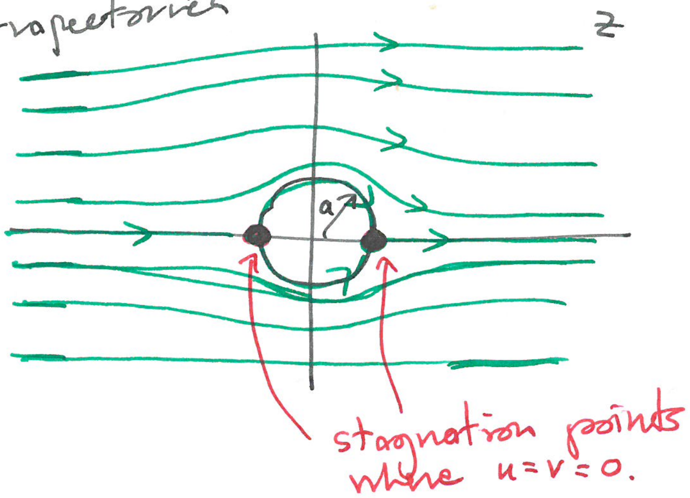
$$u - iv = U\left(1 - \frac{a^2}{z^2}\right) \longrightarrow U \text{ as } |z| \to \infty$$

$$p = \frac{\rho}{2}\left|1 - \frac{a^2}{z^2}\right| + \text{const.}$$

When $|z| = a$, $\frac{a^2}{z} = \bar{z}$, and so

$$\phi + i\psi = U(z + \bar{z}) = 2Uy.$$

In particular, $\psi = 0$ on $|z| = a$, so $|z| = a$ is made of particle trajectories



Example ~~#~~3 (Irrotational vortex)

consider $\phi + i\psi = \frac{\Gamma}{2\pi i} \log z$ on $|z| > 1$. not single-valued!

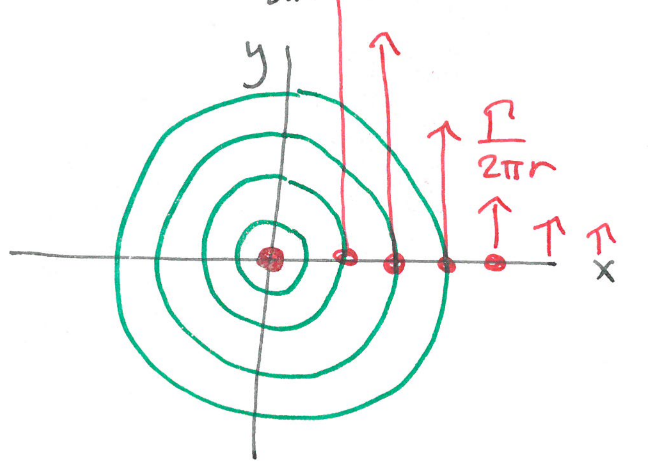
Then
$$u - iv = \frac{\Gamma}{2\pi i z} \longrightarrow 0 \text{ as } |z| \to \infty,$$
single-valued!

$$p = \frac{\rho \Gamma^2}{8\pi^2} \frac{1}{|z|^2} + \text{const.}$$

Since $\psi = \mathrm{Im} \frac{\Gamma}{2\pi i} \log z$
$$= \frac{\Gamma}{2\pi} \mathrm{Im}\left(\frac{\log r + i\theta}{i}\right)$$
$$= -\frac{\Gamma}{2\pi} \log r,$$

the streamlines are all circles $r = \text{const}$, with constant angular velocity

$$|u - iv| = \frac{\Gamma}{2\pi r}$$



Note: the motion seems "rotational" in that particles have periodic orbits. On the other hand the vorticity
$$\omega = v_x - u_y \equiv 0 \text{ for } r > 0.$$

[Explanation: $\omega = \delta_{z=0}$]

Example 4 Linear combination of Examples 2 & 3:

$$\phi + i\psi = U\left(z + \frac{a^2}{z}\right) + \frac{\Gamma}{2\pi i}\log z,$$

$|z| > a$.

~~Note that,~~

still have

$$u - iv = U\left(1 - \frac{a^2}{z^2}\right) + \frac{\Gamma}{2\pi i z}$$

$$\to U \text{ as } |z| \to \infty,$$

Also, on $|z| = a$

$$\psi = \mathrm{Im}\, U\left(z + \frac{a^2}{z}\right) + \mathrm{Im}\,\frac{\Gamma}{2\pi i}\log z$$

$$= 0 - \frac{\Gamma}{2\pi}\log a$$

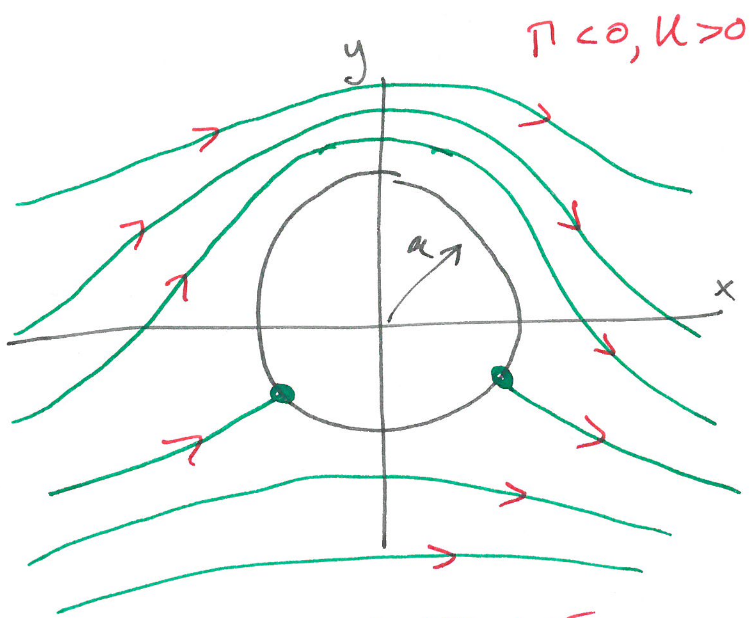
$$= \underline{\text{constant}},$$

so $|z| = a$ is still made up of streamlines.

In general,

$$\psi = U\left(y - \frac{a^2 y}{x^2 + y^2}\right) - \frac{\Gamma}{2\pi}\log r$$

$$= U\sin\theta\left(r - \frac{a^2}{r}\right) - \frac{\Gamma}{2\pi}\log r$$



IMAGINE WHAT WE COULD DO WITH CONFORMAL MAPPINGS.

Circulation

In general, the circulation around a closed loop C is

$$\oint_C \vec{u} \cdot d\vec{l} = \oint_C u_i\, dx_i$$

You guessed it, we're going to differentiate this.

Let $C^t = \Phi^t(C^0)$, and suppose that u solves the incompressible Euler eqns with $\rho \equiv$ const. Then we have

Kelvin's circulation thm

$$\frac{d}{dt}\oint_{C^t} \vec{u} \cdot d\vec{l} = 0$$

Pf Suppose that C^0 is parametrize by $\gamma : [0,1] \to \mathbb{R}^n$. Then $\Phi^t \circ \gamma$ is a parametrization of C^t, and so

$$\oint_{C^t} \vec{u} \cdot d\vec{l} = \int_0^1 \vec{u}(\Phi^t \circ \gamma(s), t) \cdot \partial_s(\Phi^t \circ \gamma)(s)\, ds.$$

Thus

$$\frac{d}{dt}\oint_{C^t} \vec{u} \cdot d\vec{l}$$

$$= \frac{d}{dt}\int_0^1 u(\Phi^t \circ \gamma, t) \cdot \partial_s(\Phi^t \circ \gamma)(s)\, ds$$

$$= \int_0^1 \frac{Du}{Dt}(\Phi^t \circ \gamma, t) \cdot \partial_s(\Phi^t \circ \gamma)(s)\, ds$$

$$+ \int_0^1 u(\Phi^t \circ \gamma, t) \cdot \partial_s \underbrace{\partial_t(\Phi^t \circ \gamma)(s)}_{= u(\Phi^t \circ \gamma, t)}\, ds$$

$$= \mathrm{I} + \mathrm{II}$$

So

$$\mathrm{II} = \int_0^1 u(\Phi^t\circ\gamma,t)\cdot\partial_s u(\Phi^t\circ\gamma,t)\,ds$$

$$= \frac{1}{2}\int_0^1 \partial_s |u|^2(\Phi^t\circ\gamma,t)\,ds$$

$$= 0.$$

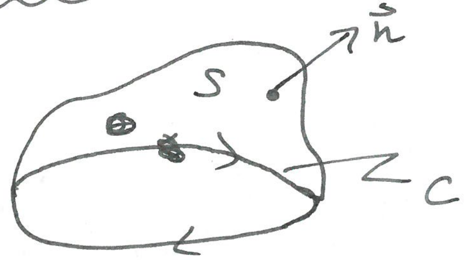
On the other hand by ☆

$$\mathrm{I} = \int_0^1 -\frac{\nabla p}{\rho}(\Phi^t\circ\gamma,t)\,\partial_s(\Phi^t\circ\gamma)\,ds$$

$$= -\frac{1}{\rho}\int_{C^t} \vec{\nabla} p\cdot d\vec{l}$$

$$= 0. \qquad \square$$

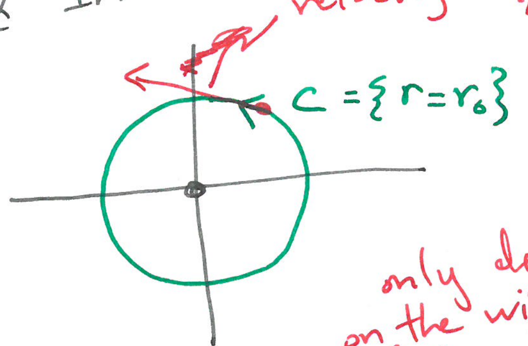
Circulation and vorticity



Stokes thm $\Rightarrow \int_C \vec{u}\cdot d\vec{l} = \int_S \vec{\omega}\cdot n\,dS.$

In particular, when $\vec{\omega}\equiv 0$, $\int_C \vec{u}\cdot d\vec{l}$ only depends on topological features of C.

Ex Irrotational vortex velocity $\frac{\Gamma}{2\pi r}$



only depends on the winding # of C

Circulation is

$$\int_C \vec{u}\cdot d\vec{l} = \frac{\Gamma}{2\pi r}\cdot 2\pi r = \boxed{\Gamma}.$$

In this sense (and other's), we have

$$\omega = \delta_{z=0}\, e_3$$

Convention for ω in 2D

Consider a 2D vector field $u:\mathbb{R}^2\to\mathbb{R}^2$. We can extend this to a 3D vector field

$$\tilde{u}(x_1,x_2,x_3) = \begin{pmatrix} u_1(x_1,x_2)\\ u_2(x_1,x_2)\\ 0\end{pmatrix}.$$

The associated vorticity is

$$\tilde{\omega} = \nabla\times\tilde{u} = \begin{vmatrix} e_1 & e_2 & e_3\\ \partial_1 & \partial_2 & \partial_3\\ u_1 & u_2 & 0\end{vmatrix}$$

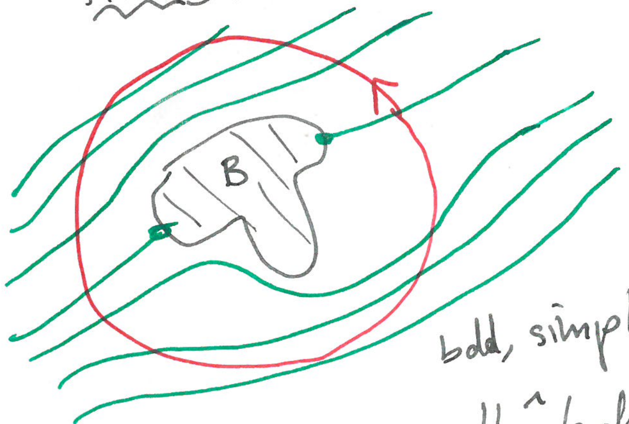
$$= \begin{pmatrix} 0\\ 0\\ \partial_1 u_2 - \partial_2 u_1\end{pmatrix}.$$

Abusing notation, we call

$$\omega = \partial_1 u_2 - \partial_2 u_1 =: \operatorname{curl} u$$

the scalar vorticity.

Circulation in 2D potential flow around a single body



Suppose B is a bdd, simply-connected smooth body ~~connected~~ ~~with $C\setminus B$~~. $u - iv$ a steady potential flow which is tangent to ∂B, with $u - iv \to u_\infty - i v_\infty \in \mathbb{C}$ as $|z|\to\infty$.

Then for any C which winds once around B,

$$\int_C \vec{u}\cdot d\vec{\ell} = \int_C (u-iv)\,dz$$

can be calculated using the calculus of residues.

<u>Pf</u> By homotopy invariance,$^{(\#1)}$ we can assume first that $C = \partial B$.

Then $\vec{u}\cdot\vec{n} = 0$ on ∂B implies

$$v\,dx = u\,dy \quad \text{on } \partial B$$

and so

$$\int_C (u-iv)\,dz = \int_C (u-iv)(dx+i\,dy)$$

$$= \underbrace{\int_C (u\,dx + v\,dy)}_{=\int_C \vec{u}\cdot d\vec{\ell}} + i\int \underbrace{\cancel{(u\,dy - v\,dx)}}_{\to 0}$$

$$= \int_C \vec{u}\cdot d\vec{\ell}.$$

By homotopy invariance$^{(\#2)}$, for any C we have

$$\int_C \vec{u}\cdot d\vec{\ell} = \int_C (u-iv)\,dz$$

$$= \int_{\{r=r_0\}} (u-iv)\,dz.$$

This last integral can be calculated in terms of the Laurent series for $u-iv$,

$$u-iv = u_\infty - iv_\infty + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

for $|z| \geq r_0 \gg 1$.

we get

$$\int_C \vec{u}\cdot d\vec{\ell} = \int_{\{r=r_0\}} (u-iv)\,dz$$

$$= \int_{\{r=r_0\}} \left(u_\infty - iv_\infty + \frac{a_1}{z} + \cdots\right)dz$$

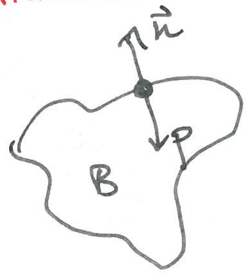
$$= 2\pi i a_1. \qquad \square$$

Kutta-Joukowski Thm

For a body B and $u-iv$ as above, the force exerted on the body by the fluid is

$$\rho\,(v_\infty - iu_\infty)\underbrace{\Gamma}_{\text{circulation around } B} \in \mathbb{C} \cong \mathbb{R}^2$$

<u>Pf</u> We know that



$$\text{Force on } B = -\text{Force on fluid}$$

$$= -\int_{\partial B} p\,n\,d\ell.$$

On the other hand by Bernoulli,

$$p = -\frac{\rho}{2}\overbrace{|u-iv|^2}^{u^2+v^2} + \text{const.}$$

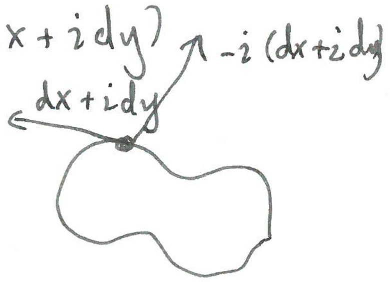
Since $\displaystyle\int_{\partial B} \text{const.}\; n\,d\ell = \int_B \nabla\,\text{const} = 0,$

assume const. $= 0$ WLOG.

Now with $\mathbb{R}^2 \cong \mathbb{C}$,

$$n\,dl = -i(dx + i\,dy)$$

and so



$$\text{Force on } B = -\int_{\partial B} p\, n\, dl$$

$$= \int_{\partial B} \rho \frac{(u^2+v^2)}{2}\, n\, dl$$

$$= -\frac{i\rho}{2}\int_{\partial B} (u^2+v^2)\,dz.$$

Claim $\int_{\partial B} (u^2+v^2)\,dz = \overline{\int_{\partial B} (u-iv)^2\,dz}$

Pf of Claim

$$\text{LHS} - \text{RHS} = \cdots = -2i\int (u+iv)\overset{\substack{0\\ \shortparallel}}{\boxed{(v\,dx - u\,dy)}}$$

$$= 0. \quad \square$$

As before, can calculate $\int_{\partial B} (u-iv)^2\,dz$ using residues,

$$\int_{\partial B} (u-iv)^2\,dz = \int_{\{r=r_0\}} (u-iv)^2\,dz$$

$$= \int_{\{r=r_0\}} \left(u_\infty - iv_\infty + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \right)^2 dz$$

$$= \int_{\{r=r_0\}} \left((u_\infty - iv_\infty)^2 + \frac{2(u_\infty - iv_\infty)a_1}{z} + \cdots \right) dz$$

$$= 2\pi i \cdot 2(u_\infty - iv_\infty)a_1$$

$$= 2(u_\infty - iv_\infty)\Gamma.$$

Thus

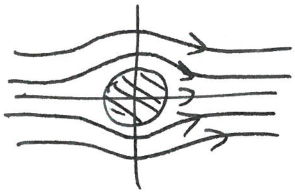
$$\text{Force on } B = -\frac{i\rho}{2}\,\overline{2(u_\infty - iv_\infty)\Gamma}$$

$$= \rho\Gamma(-i)(u_\infty + iv_\infty)$$

$$= \boxed{\rho\Gamma(v_\infty - iu_\infty)}. \quad \square$$

Application to our examples

Ex 2



$$\phi + i\psi = \underset{\in\mathbb{R}}{U}\left(z + \frac{a^2}{z}\right)$$

$$u - iv = U\left(1 - \frac{a^2}{z^2}\right)$$

$$u_\infty - iv_\infty = U \in \mathbb{R},$$

$$\text{circulation } \Gamma = \int_C U\left(1 - \frac{a^2}{z^2}\right)dz = 0.$$

∴ Force on $B = 0$.

Ex 3 $\phi + i\psi = \frac{\Gamma}{2\pi i}\log z$

$$u - iv = \frac{\Gamma}{2\pi i z}, \quad u_\infty - iv_\infty = 0$$

$$\text{circulation } \Gamma = \int_C \frac{\Gamma}{2\pi i z}\,dz = \Gamma \ \checkmark$$

∴ Force on $B = 0$.