DAY 08

Basic fact about $\mathbb{P}_\lambda$.

Cor 2.2 For $u \in \mathcal{H}$, $\mathbb{P}_\lambda u \in \mathcal{V}_\sigma$
with $|\nabla \mathbb{P}_\lambda u|_{L^2} \leq \sqrt{\lambda}\, |u|_{L^2}$.
For $u \in \mathcal{V}_\sigma$, $|(I - \mathbb{P}_\lambda)u|_{L^2} \leq \frac{1}{\sqrt{\lambda}} |u|_{\mathcal{V}_\sigma}$

Pf Let $n(\lambda) = \sup\{j : \mu_j^2 < \lambda\}$.
Then

$$|\nabla \mathbb{P}_\lambda u|_{L^2}^2 = \Big| \sum_{j<n} \langle u, e_j \rangle \nabla e_j \Big|_{L^2}^2$$

$$\text{(ONB)} \quad = \sum_{j<n} \langle u, e_j \rangle^2 \underbrace{|\nabla e_j|_{L^2}^2}$$

$$= \langle -\Delta e_j, e_j \rangle$$

$$= \mu_j^2 |e_j|_{L^2}$$

$$= \mu_j^2$$

$$\leq \lambda \sum_{j=1}^{\infty} \langle u, e_j \rangle^2$$

$$= \lambda |u|_{L^2}^2 .$$

Similarly

$$|(I - \mathbb{P}_\lambda)u|_{L^2}^2 = \sum_{j \geq n} \langle u, e_j \rangle^2$$

$$|u|_{\mathcal{V}_\sigma}^2 = \sum_j \langle u, e_j \rangle^2 \mu_j^2$$

$$\geq \sum_{j \geq n} \langle u, e_j \rangle^2 \mu_j^2$$

$$\geq \lambda \sum_{j \geq n} \langle u, e_j \rangle^2 . \quad \square$$

DAY 08

Recall

$$(NS)\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = f \\ \nabla\cdot u = 0 \\ u|_{\partial\Omega} = 0 \\ u|_{t=0} = u_0 \end{cases}$$

Energy (in)equality

$$\frac{1}{2}\int |u|^2 dx + \nu\int_0^t\int |\nabla u|^2 dx\,dt' \overset{(\leq)}{=} \frac{1}{2}\int |u_0|^2 dx + \int_0^t\int f\cdot u\,dx\,dt'$$

$\mathcal{D} = C_0^\infty(\Omega)$, $\mathcal{D}' =$ distributions

$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathcal{D}'\times\mathcal{D}}$

$H_0^1 = \overline{\mathcal{D}}^{H^1}$, $(u,v)_{H_0^1} = (\nabla u,\nabla v)_{L^2}$

$H^{-1} = (H_0^1)'$

$\mathcal{V} = (H_0^1)^3$, $\mathcal{V}_\sigma = \mathcal{V}\cap\{\nabla\cdot v = 0\}$

$\mathcal{V}' = (H^{-1})^3$, $\mathcal{H} = \overline{\mathcal{V}_\sigma}^{L^2}$

$\mathcal{V}_\sigma^0 = \{f\in\mathcal{V}' : \langle f,v\rangle = 0 \ \forall v\in\mathcal{V}_\sigma\}$
$= \{\nabla\pi\in\mathcal{V}_\sigma^0 : \pi\in L^2_{loc}\}$

$A:\mathcal{V}_\sigma\to\mathcal{V}'$, $Au = f \Leftrightarrow -\Delta u - f\in\mathcal{V}_\sigma^0$

$\{e_k\}$ ONB of $\mathcal{H}$, $Ae_k = \mu_k^2 e_k$

$\{\mu_k^{-1}e_k\}$ ONB of $\mathcal{V}_\sigma$

$\mathbb{P}_\lambda : \mathcal{V}'\to\mathcal{V}_\sigma$, $\mathbb{P}_\lambda f = \sum_{\{j:\mu_j^2\leq\lambda\}}\langle f,e_j\rangle e_j$

$\mathbb{P}_\lambda f\to\mathbb{P}f$ for $f\in(L^2)^d$

$\mathbb{P} : (L^2)^d\to\mathcal{H}$ orthogonal proj.

Prop 2.3 $\tilde{\mathbb{P}}_\lambda : \mathcal{V}'\to\mathcal{V}_\sigma'$ defined by $\langle\tilde{\mathbb{P}}_\lambda f, v\rangle = \langle f,\mathbb{P}_\lambda v\rangle$ satisfies

(a) $|\tilde{\mathbb{P}}_\lambda f|_{\mathcal{V}_\sigma'}\leq|f|_{\mathcal{V}_\sigma'}$

(b) $\forall f\in\mathcal{V}_\sigma'$, $\tilde{\mathbb{P}}_\lambda f\in\mathcal{V}_\sigma$ and $\tilde{\mathbb{P}}_\lambda f\to f$ in $\mathcal{V}_\sigma'$

Pf "Follow your nose." Use prop 1.1. □

ABUSE OF NOTATION: WRITE $\tilde{\mathbb{P}}_\lambda$ AS $\mathbb{P}_\lambda$.

§2.1 The Leray Thm

Def u is a global weak soln of (NS) with $f\in L^2_{loc}\mathcal{V}'$ if

$u\in C\mathcal{V}_\sigma'\cap L^\infty_{loc}\mathcal{H}\cap L^2_{loc}\mathcal{V}_\sigma$ $:= C(\mathbb{R}^+,\mathcal{V}_\sigma')$ etc.

and $\forall\Phi\in C^1\mathcal{V}_\sigma$ we have

$$(S_\Phi)\quad \int u\cdot\Phi\,dx + \int_0^t\int\Big(\nu\nabla u:\nabla\Phi - u\otimes u:\nabla\Phi - u\cdot\partial_t\Phi\Big)dx\,dt' = \int u_0\cdot\Phi\,dx + \int_0^t\langle f,\Phi\rangle dt'$$

Thm (Leray) $\forall u_0\in\mathcal{H}$ $\exists$ a global weak soln of (NS) which satisfies the energy inequality.

Def Such solutions are called Leray solutions.

Prop 2.5 $\exists C = C(\Omega)$ s.t. Leray solns satisfy

$$|u|_{L^2}^2 + \nu\int_0^t|\nabla u|_{L^2}^2 dt' \leq |u_0|_{L^2}^2 + \frac{C}{\nu}\int_0^t|f|_{\mathcal{V}_\sigma'}^2 dt'$$

Pf Use energy inequality and

$$\langle f,u\rangle\leq|f|_{\mathcal{V}_\sigma'}|u|_{\mathcal{V}} = |f|_{\mathcal{V}_\sigma'}|\nabla u|_{L^2} \leq \nu|\nabla u|_{L^2}^2 + \frac{C}{\nu}|f|_{\mathcal{V}_\sigma'}^2.$$

□

PLAN OF PROOF

1. Use $\mathbb{P}_k$, $k\in\mathbb{N}$, to define an approximate equation (NS_k) which is well-behaved.
2. Derive bounds on u_k that are uniform in k ⇝ compactness.
3. Take a limit in (S_Φ) as $k\to\infty$, paying special attention to the nonlinear term

$$\int_0^t\int_\Omega u\otimes u:\nabla\Phi\,dx\,dt'.$$

§ 2.2.1 Approximate solns

Def • $\mathcal{H}_k = \mathbb{P}_k \mathcal{H}$, $\boxed{k \in \mathbb{N}}$
$= \operatorname{span}\{e_1, \dots, e_{n(k)}\}$

Lem 2.2 $\forall f \in L^2_{loc} \mathcal{V}'$ $\exists f_k \in C^1 \mathcal{V}_\sigma'$ s.t. $f_k(t) \in \mathcal{H}_k$ $\forall t$ and $f_k \to f$ in $L^2([0,T], \mathcal{V}_\sigma')$ $\forall T$.

Pf Prop 2.3 + time-regularization

The main thing we are worried about is the nonlinear term $u \otimes u : \nabla \psi$ or $-\operatorname{div}(u \otimes u) \cdot \psi$.

Def $Q : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}'$
$$Q(u,v) = -\nabla \cdot (u \otimes v)$$

Lem 2.3 For $d = 2,3$ $\exists C > 0$ s.t.
$$\langle Q(u,v), \varphi \rangle \leq C |\nabla u|_{L^2}^{d/4} |\nabla v|_{L^2}^{d/4} |u|_{L^2}^{1-d/4} |v|_{L^2}^{1-d/4} |\nabla \varphi|_{L^2}$$
for all $u, v, \varphi \in \mathcal{V}$. Moreover, $u \in \mathcal{V}_\sigma$ and $v \in \mathcal{V} \Rightarrow \langle Q(u,v), v \rangle = 0$.

Pf For the inequality,
$$\langle Q(u,v), \varphi \rangle \leq |u \otimes v|_{L^2} |\nabla \varphi|_{L^2}$$
$$\leq \underbrace{|u|_{L^4} |v|_{L^4}} |\nabla \varphi|_{L^2}$$
now apply Gagliardo-Nirenberg.

For the identity, by density it is enough to prove when u, v are C_c^∞. Then we can IBP to get

$$\langle Q(u,v), v \rangle = -\int \operatorname{div}(u \otimes v) \cdot v \, dx$$
$$= -\int \partial_i (u_i v_j) v_j \, dx$$
$$= -\int \cancel{(\partial_i u_i)}^{\,0} v_j v_j \, dx$$
$$- \int u_i (\partial_i v_j) v_j \, dx$$
$$= + \int v_j \partial_i (u_i v_j) \, dx$$
$$= -\langle Q(u,v), v \rangle$$
and hence $\langle Q(u,v), v \rangle = 0$. □

Def (Approximate nonlinearity)
$$F_k(u) := \mathbb{P}_k Q(u,u)$$

$$\left[\begin{array}{ccccccc} u & \longmapsto & (u,u) & \longmapsto & Q(u,u) & \longmapsto & \mathbb{P}_k Q(u,u) \\ \mathcal{V} & \to & \mathcal{V} \times \mathcal{V} & \longrightarrow & \mathcal{V}' & \longrightarrow & \mathcal{H}_k \end{array} \right]$$

$$(NS_k) \begin{cases} u_k : \mathbb{R}^+ \longrightarrow \mathcal{H}_k \\ \dot{u}_k = \nu \mathbb{P}_k \Delta u_k + F_k(u_k) + f_k \\ u_k|_{t=0} = \mathbb{P}_k u_0 \end{cases}$$

By definition of $\mathbb{P}_k$, $\mathbb{P}_k \Delta$ is bdd $\mathcal{H}_k \to \mathcal{H}_k$. Moreover F_k is locally Lipschitz (EXERCISE). So there exists a *unique* soln to (NS_k) in $C^\infty([0,T_k), \mathcal{H}_k)$ for some maximal time $T_k \leq +\infty$. Want to show $T_k = +\infty$.

ODEs in $\mathbb{R}^n$ or in Banach spaces

[Formally, $(NS_k) \longrightarrow (NS)$ as $k \to \infty$.]

§2.2.2 A proiri bounds

Prop 2.6 The sequence u_k is bdd in

$$L^\infty_{loc} H \cap L^2_{loc} V_\sigma \cap L^{8/d}_{loc} L^4$$

Pf (Energy estimate)

Testing (NS_k) against u_k we get

$$\frac{1}{2}\frac{d}{dt}|u_k|^2_{L^2} = \nu(\Delta u_k, u_k)_{L^2} + (F_k(u_k), u_k)_{L^2} + (f_k, u_k)_{L^2}.$$

But

$$\begin{aligned}(F_k(u_k), u_k)_{L^2} &= (\mathbb{P}_k Q(u_k,u_k), u_k)_{L^2} \\ &= \langle Q(u_k,u_k), \mathbb{P}_k u_k\rangle \\ &= \langle Q(u_k,u_k), u_k\rangle \\ &= 0 \qquad (\text{Lem } 2.3).\end{aligned}$$

Integrating dt, get

$$\begin{aligned}\frac{1}{2}|u_k|^2_{L^2} + \nu\int_0^t |\nabla u_k|^2_{L^2}\,dt' &= \frac{1}{2}|u_k(0)|^2_{L^2} + \int_0^t (f_k,u_k)_{L^2}\,dt' \\ &\le \frac{1}{2}|u_k(0)|^2_{L^2} + \int_0^t |f_k|_{V_\sigma'}|u_k|_{V_\sigma}\,dt' \\ &\le \frac{1}{2}|u_k(0)|^2_{L^2} + \frac{C}{\nu}\int_0^t |f_k|^2_{V_\sigma'}\,dt \\ (\text{poincaré}) \quad & \quad + \frac{\nu}{2}\int_0^t |\nabla u_k|^2_{L^2}\,dt'\end{aligned}$$

ABSORB INTO LHS

and hence

$$\frac{1}{2}|u_k|^2_{L^2} + \frac{\nu}{2}\int_0^t |\nabla u_k|^2_{L^2}\,dt' \le \frac{1}{2}|u_k(0)|^2_{L^2} + \int_0^t |f_k|^2_{V_\sigma'}\,dt'$$

In particular, $|u_k|_{L^2} = |u_k|_{H_k}$ cannot blow up, and so $T_k = +\infty$. But even more than that, we have

$$\begin{aligned}&|u_k|_{L^\infty([0,T],L^2)} + |u_k|_{L^2([0,T];V_\sigma)} \\ &\le C\big(|u_k(0)|_{L^2} + |f_k|_{L^2([0,T],V_\sigma')}\big) \\ &\to C\big(|u_0|_{L^2} + |f|_{L^2([0,T],V_\sigma')}\big)\end{aligned}$$

and so $\{u_k\}$ is bdd in

$$L^\infty_{loc} L^2 \cap L^2_{loc} V_\sigma.$$

Using Gagliardo - Nirenburg to interpolate b/w L^2 & V_σ, get bddness in $L^{8/d}_{loc} L^4$ also. $\square$

Also note that for $v \in V_\sigma$,

$$\langle -\Delta u_k, v\rangle = (\nabla u_k, \nabla v)_{L^2} \le |u_k|_{H^1_0}|v|_V$$

and so

$$|\Delta u_k|_{V_\sigma'} \le |u_k|_{H^1_0} = |\nabla u_k|_{L^2}$$

and hence

$$|\Delta u_k|_{L^2([0,T],V_\sigma')} \le |\nabla u_k|_{L^2([0,T];}$$

is also uniformly bdd.

§2.2.3 Compactness

<u>Prop 2.7</u> $\exists u \in L^2_{loc} V_\sigma$ s.t., after extraction,

$$\lim_{k\to\infty} \int_{[0,T]\times K} |u_k - u|^2 \,dx\,dt = 0 \quad \circledast$$

for all $T>0$ and compact $K \subseteq \Omega$.

Moreover, for

$\Psi \in L^2([0,T], V)$
$\Phi \in L^2([0,T]\times\Omega)$
$\psi \in C^1([0,T], V_\sigma)$

we have

$$\lim_{k\to\infty} \int_{[0,T]\times\Omega} \nabla(u_k - u) : \nabla\Psi \,dx\,dt = 0$$

$$\lim_{k\to\infty} \int_{[0,T]\times\Omega} (u_k - u)\cdot\Phi \,dx\,dt = 0$$

$$\lim_{k\to\infty} \sup_{t\in[0,T]} |\langle u_k - u, \psi\rangle| = 0.$$

<u>Pf</u> By usual diagonalization procedure with a sequence $T_n \to \infty$ and $K_1 \subset K_2 \subset \cdots$ with $\bigcup K_n = \Omega$, enough to show $\circledast$ for fixed T & K.

Want to show that $\{u_k\}$ is relatively compact in $L^2([0,T]\times K)$, i.e. that it can be covered by finitely many $L^2([0,T]\times K)$ balls with radius ε for any $\varepsilon>0$.

So let $\varepsilon>0$. First we apply a <u>fixed</u> projection P_{k_0}.

~~we claim that we can~~

Then

$$|(I - P_{k_0})u_k|_{L^2} \le \frac{1}{\sqrt{k_0}} |u_k|_{V_\sigma}$$

and so

$$|(I - P_{k_0})u_k|_{L^2([0,T],L^2)} \le \frac{1}{\sqrt{k_0}} \underbrace{|u_k|_{L^2([0,T],V_\sigma)}}_{\text{uniformly bdd}}$$

In particular, we can choose k_0 large enough that

$$|(I - P_{k_0})u_k|_{L^2([0,T]\times\Omega)} \le \frac{\varepsilon}{2} \quad \forall k.$$

So it is sufficient to cover $\{P_{k_0} u_k\}$ by finitely many $\frac{\varepsilon}{2}$-balls in $L^2([0,T]\times K)$.

We claim that $\{P_{k_0} u_k\}$ is uniformly bdd in

$$C^{1-\frac{d}{4}}([0,T], V_\sigma),$$

~~and so~~

$\circledast$ then follows since $V_\sigma \hookrightarrow L^2(K)$ is compact.

First let's estimate

$$|\partial_t u_k|_{L^{\frac{4}{d}}([0,T], V_\sigma')}.$$

By (NS_k),

$$\partial_t u_k = \nu\Delta u_k + F_k(u_k) + f_k,$$

and $\Delta u_k, f_k$ are unif. bdd in $L^2([0,T], V_\sigma')$. For F_k we estimate

$$|F_k|_{V_\sigma'} \le C |\nabla u_k|_{L^2}^{\frac{d}{2}} |u_k|_{L^2}^{2-\frac{d}{2}}$$
$$\le C |\nabla u_k|_{L^2}^{d/2}$$

and so

$$|F_k|_{L^{4/d}([0,T], V_\sigma')} \le C \left(|\nabla u_k|^2_{L^2 L^2} \right)^{d/4}$$

is uniformly bdd also. Thus

$$|\partial_t u_k|_{L^{4/d}([0,T], V_\sigma')}$$

is unif. bdd. $\left(\frac{4}{d} \le 2\right)$

By Cor 2.2,

$$|\partial_t \mathbb{P}_{k_0} u_k|_{L^{4/d}([0,T], V_\sigma)}$$

is then also uniformly bdd.
Thus $\{\mathbb{P}_{k_0} u_k\}$ is uniformly bdd in

$$C^{1-\frac{d}{4}}([0,T], V_\sigma).$$

Since $V_\sigma \hookrightarrow L^2(K)$ is compact, we conclude that $\{\mathbb{P}_{k_0} u_k\}$ is compact in $C^0([0,T], L^2(K))$, and hence in $L^2([0,T] \times K)$, as desired.

[Skipping rest of proof.] $\square$

§ 2.2.4 End of proof of Leray thm

Let $\Psi \in C^1 V_\sigma$ as in (S_Ψ).
Integrating by parts we find

$$\int u_k \cdot \Psi \, dx + \int_0^t \int_\Omega \left(\begin{array}{l} \nabla u_k : \nabla \mathbb{P}_k \psi \\ - u_k \otimes u_k : \nabla \mathbb{P}_k \psi \\ - u_k \cdot \partial_t \psi \end{array} \right) dx\, dt' \qquad (S_{\Psi,k})$$

$$= \int u_k \cdot \Psi |_{t=0} \, dx + \int_0^t \langle f_k, \psi \rangle dt'$$

We want to pass to the limit. To deal with the $\mathbb{P}_k$'s we use

<u>Lem 2.4</u> Let H be a Hilbert space and $A_n : H \to H$ a bdd sequence of operators which converge strongly to the identity in that

$$A_n h \longrightarrow h \quad \forall h \in H.$$

Then for $\psi \in C([0,T], H)$ we have

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \| A_n \psi(t) - \psi(t) \|_H = 0$$

<u>Pf</u> Easy. $\square$

So $\lim_{k \to \infty} \sup_{t \in [0,T]} \| \mathbb{P}_k \Psi - \Psi \|_V = 0.$ (send $k \to \infty$)

This and Prop 2.7 lets us <u>deal with</u> all of the linear terms in $(S_{\Psi,k})$. It remains to show that

$$\lim_{k \to \infty} \int_0^t \int_\Omega u_k \otimes u_k : \nabla \Psi \, dx\, dt'$$
$$= \int_0^t \int_\Omega u \otimes u : \nabla \Psi \, dx\, dt'.$$

Let $K_1 \subset K_2 \subset \cdots$, $\cup K_n = \Omega$ as before. Then Lem 2.4 implies

$$1_{K_n}\nabla\Psi \longrightarrow \nabla\Psi \quad \text{in } L^\infty([0,T], L^2)$$

Thus we just need to show

$$u_k \otimes u_k \longrightarrow u \otimes u \quad \text{in } L^1([0,T], L^2(K))$$

which is implied by

$$u_k \longrightarrow u \quad \text{in } L^2([0,T], L^4(K)).$$

Now

$$|u_k - u|_{L^4(K)} \le C\,|u_k - u|_{L^2_K}^{1-\frac{d}{4}}\,|\nabla(u_k - u)|_{L^2}^{d/4}$$

and so

$$|u_k - u|_{L^2([0,T], L^4(K))} \le C\,\underbrace{|u_k - u|^{1-\frac{d}{4}}_{L^2([0,T]\times K)}}_{\to 0}\,\underbrace{|\nabla(u_k - u)|^{d/4}_{L^2([0,T]\times\Omega)}}_{\text{bdd}}$$

So u is a global weak soln!

Now to show the energy ineq. We have shown that

$u_k(t) \to u(t)$ weakly in $\mathcal{H}$

$u_k \to u$ weakly in $L^2_{loc}V$

Thus

$$|u(t)|_{L^2} \le \liminf_{k\to\infty} |u_k(t)|_{L^2}$$

$$\int_0^t |\nabla u|^2_{L^2}\,dt' \le \liminf_{k\to\infty} \int_0^t |\nabla u_k|^2_{L^2}\,dt',$$

Using this in the energy <u>equality</u> for (NS_k) we get the energy <u>inequality</u> ~~for (NS~~ for (NS).

Finally, can check that $u \in C^0(\mathbb{R}^+, V'_\sigma)$ by using (S_Ψ) when $\Psi = \Psi(x)$ does not depend on t.