1

DAY 04

STILL FOCUSED ON THE INCOMP. EULER EQNS W/CONST DENSITY:

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p \quad \cancel{+ f} \\ \nabla \cdot u = 0 \end{cases}$$

~~external forces~~

TODAY EVERYTHING WILL BE IN $n=2$ DIMENSIONS.

Last time we saw that the assumptions $\omega = \nabla \times u \equiv 0$, and $\frac{\partial u}{\partial t} \equiv 0$ simplify things considerably. Today we will drop these assumptions and look at the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p \\ \nabla \cdot u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

Cauchy data

The vorticity ω will play a prominent role.

WHAT ARE SOME MORE FAMILIAR CAUCHY PROBLEMS?

The vorticity eqn in 2D

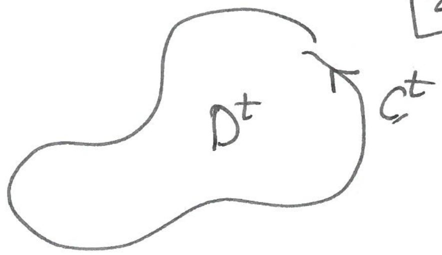
Consider a domain $D^t = \Phi^t(D^0)$ with boundary $C^t = \partial D^t = \Phi^t(C^0)$. By Kelvin's circulation theorem

$$\frac{d}{dt} \int_{C^t} u \cdot dl = 0.$$

On the other hand by Green's theorem

$$\int_{C^t} u \cdot dl = \int_{D^t} \omega \, dx$$

2



Thus

$$\frac{d}{dt} \int_{D^t} \omega \, dx = 0.$$

On the other hand we have our usual chain rule formula:

$$\frac{d}{dt} \int_{D^t} \omega \, dx = \int_{D^t} \left(\frac{D\omega}{Dt} + \omega \underbrace{\cancel{(\nabla \cdot u)}}_{0} \right) dx$$

and so we get

$$\int_{D^t} \frac{D\omega}{Dt} \, dx = 0.$$

Since D^t was arbitrary,

$$\boxed{\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega = 0}$$

This means that

$\boxed{\omega \text{ is constant along particle trajectories}}$

This is ~~tremendously~~ extremely useful and only valid in 2D.

Many arguments follow the rough outline:

$$\underbrace{\omega \longmapsto u = \text{"curl}^{-1}\omega\text{"} \longmapsto \Phi^t, \text{ particle trajectories} \longmapsto \text{new } \omega}$$

FIND A FIXED POINT

Some background

Fixed-point theorems

X is a Banach space

Banach: If $F: X \to X$ has $|F(x)-F(y)| < \rho|x-y|$ for some fixed $\rho < 1$, then F has a unique fixed point.

unit ball $\subseteq \mathbb{R}^n$

Brouwer If $f: B \to B$ is continuous then it has at least one fixed point.

Schauder If $K \subseteq X$ is convex and compact, then every continuous $F: K \to K$ has at least one fixed point.

Idea: if K were finite dimensional then this is Brouwer. But compactness means it is "almost" finite dimensional.

Recovering u from ω

Since we're in 2D and $\nabla \cdot u = 0$, we know that

$$u = \nabla^\perp \psi =: (\partial_2 \psi, -\partial_1 \psi)$$

for some stream function ψ.

So

$$\omega = \text{"}\nabla \times u\text{"} = -\nabla^\perp \cdot u$$
$$= -\nabla^\perp \cdot \nabla^\perp \psi$$
$$= -\Delta \psi.$$

If we could invert $-\Delta$, then this would give

$$\psi = (-\Delta)^{-1} \omega$$
$$u = \nabla^\perp \psi = \nabla^\perp (-\Delta)^{-1} \omega$$

This brings us to

Basic potential theory (2D!)

If a smooth function f has compact support in $\mathbb{R}^2$ then

$$f(x) = \int_{\mathbb{R}^2} \frac{1}{2\pi} \log|x-y| \, \Delta f(y) \, dy$$
$$= \underbrace{\frac{1}{2\pi} \log|\cdot|}_{\text{"fundamental solution"}} * \Delta f$$

For a smooth bdd domain Ω, the problem

$$\begin{cases} -\Delta f = g & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega \end{cases}$$

similarly has a unique soln, expressible as

$$f(x) = \int_\Omega \boxed{G^\Omega(x,y)} \, g(y) \, dy$$

(*)

Green's function for $-\Delta$ on Ω

G^Ω is smooth away from $x = y$, and near $x = y$ it looks like

$$G^\Omega(x,y) = -\frac{1}{2\pi} \log|x-y| + \cancel{\text{smaller}} \text{ smooth terms}$$

We write (*) as

$$f = (-\Delta)^{-1} g.$$

Hölder spaces

Unfortunately, while
$$u \in C^{k+2} \Longrightarrow \Delta u \in C^k$$
for $k = 0, 1, 2, \dots$, the reverse implication is NOT TRUE. In particular,
$$u = (x_1^2 - x_2^2)(-\log|x|)^{1/2} \notin C^2$$
while
$$\Delta u = \text{ugly} \;\text{-----}\; \in C^0.$$

So we need different spaces of functions which somehow have "fractional" regularity.

big list/tensor of all k-th order partials

Def If $u, Du, \dots, D^k u$ exist and are continuous on Ω we say $u \in C^k(\Omega)$. If they continuously extend to $\bar\Omega$ then we say $u \in C^k(\bar\Omega)$. $C^k(\bar\Omega)$ is a Banach space with the norm
$$|u|_{C^k(\bar\Omega)} = \sum_{j=0}^{k} \sup_{\Omega} |D^j u|$$
$\left(\text{''} \; |u|_{C^k} \right)$

Hölder semi-norm

Def For $0 < \theta < 1$ we set
$$[u]_{C^\theta(\Omega)} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\theta}$$

When $[u]_{C^\theta} < \infty$ we say that u is Hölder continuous with exponent θ. $\boxed{\text{Ex: } u(x) = |x|^\theta}$

Def For $k = 0, 1, 2, \dots$ and $\theta \in (0,1)$ we set
$$|u|_{C^{k+\theta}} = |u|_{C^k} + [D^k u]_{C^\theta}.$$

The corresponding Banach space is called $\boxed{C^{k+\theta}(\bar\Omega)}$. $\boxed{\text{EX: } x^{k+\theta}, \; x \geq 0}$

Fact For any $f \in C^{k+\theta}(\bar\Omega)$, the problem
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$
has a unique solution $u \in C^{k+2+\theta}(\bar\Omega)$. The corresponding linear map $(-\Delta)^{-1} : u \longmapsto f$ is bdd $C^{k+\theta}(\bar\Omega) \to C^{k+2+\theta}(\bar\Omega)$

(Gilbarg-Trudinger Cor 4.14)

SO, these are good spaces to study $(-\Delta)^{-1}$ in, and hence good spaces to use when we want to use formulas like
$$\text{''} \; u = \nabla^\perp (-\Delta)^{-1} \omega \; \text{''}$$

- Kato 1967: On classical solutions of the Two-Dim. Non-stationary Euler Eqn

This paper is available online and is very readable. [Some of the references are bit old, and some of the notation is less fashionable]

We will change notation in a few places to match what we've been using so far.

Let Ω be a smooth bdd domain $\subseteq \mathbb{R}^2$, with boundary components $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ as above. Consider the Euler equation in $Q_T = \Omega \times [0,T]$

$$(E) \begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla p + f \\ \nabla\cdot u = 0 \end{cases} \Omega$$
$$u\cdot n = 0 \quad \text{on } \partial\Omega$$
$$u|_{t=0} = u_0$$

more on this soon!

Thm (Kato) Suppose $u_0 \in C^{1+\alpha}(\overline{\Omega})$, $\nabla\cdot u_0 = 0$, $a\cdot n = 0$ on Γ, and $\boxed{f \in C^{1+\alpha,0}(\overline{Q}_T)}$ for some Hölder exponent $\alpha\in(0,1)$. Then $\exists$ a soln (u,p) of (E) such that u,p & all their derivatives appearing in (E) are $C(\overline{Q}_T)$. Such a solution is unique up to an arbitrary function of t which may be added to p.

Simplifications We will assume that

$$\begin{cases} \Gamma = \Gamma_0 & \text{so } \Omega \text{ is simply-connected} \\ f = 0 & \text{so no external forces} \end{cases}$$

Also we will not completely prove every lemma! "Exercise for the reader"

Definitions and notation

$\nabla\cdot u = 0 \iff$ "u is a flow"

$\left.\begin{array}{l}\nabla\cdot u = 0, \\ u\cdot n = 0 \\ \text{on } \Gamma\end{array}\right\} \iff$ "u is a tangential flow"

- $(u,v) = \int_\Omega u\cdot v\,dx = (u,v)_{L^2}$

  $|u|^2_{L^2} = (u,u)$.

- $u\in C^k$ always means $u \in C^k(\overline{\Omega})$ or $C^k(\overline{Q}_T)$ and never just $u\in C^k(\Omega)$.

$u \in C^{j,k}(\overline{Q}_T)$ means

$$D_x^p D_t^q u \in C^0(\overline{Q}_T)$$

for $0\le p\le j$, $0\le q\le k$.

For $0\le\delta<1$, $0\le\varepsilon<1$, $u\in C^{j+\delta,k+\varepsilon}(\overline{Q}_T)$ means that $u\in C^{j,k}(\overline{Q}_T)$ and that

$$[D_x^p D_t^q u]_{C^{\delta,\varepsilon}} < \infty$$

for $0\le p\le j$, $0\le q\le k$, where

$$[u]_{C^{\delta,\varepsilon}} := \mathbb{1}_{\delta>0} \sup_t \sup_{x\neq y} \frac{|u(x,t)-u(y,t)|}{|x-y|^\delta} + \mathbb{1}_{\varepsilon>0} \sup_x \sup_{t\neq s} \frac{|u(x,s)-u(x,t)|}{|x-t|^\varepsilon}$$

§1 Preliminaries

Lem 1.1 Let $u\in C^1(\bar\Omega)$ be a flow and set $\omega = D\times u = -D^\perp\cdot u$. Then

$$(\varphi, u\cdot D\Phi)+((u\cdot D)u, D^\perp\Phi)=0$$

for all $\Phi\in C^1(\bar\Omega)$ ~~which~~ vanish on Γ.

$\boxed{\text{sketch}}$

Pf) When $u\in C^2(\bar\Omega)$, this can be proven by integrating by parts (divergence theorem and/or Green's theorem). For $u\in C^1(\bar\Omega)$, first extend to $\tilde u\in C^1(\tilde\Omega)$ where $\tilde\Omega\supset\bar\Omega$ and then apply to

$$u^\varepsilon = \underbrace{\phi^\varepsilon}_{\text{sequence of mollifiers approximating } \delta} * u \in C^\infty.$$

Lem 1.2 (i) If $\varphi\in C^{\delta,\varepsilon}(\bar Q_T)$ with $\delta,\varepsilon\in(0,1)$, then

$$t\longmapsto\varphi(\cdot,t)$$

is a $C^{(1-\lambda)\varepsilon}$ function

$$[0,T]\longrightarrow C^{\lambda\delta}(\bar\Omega)$$

for any $\lambda\in(0,1)$. (ii) If $\varphi\in C^{\delta,0}(\bar Q_T)$ with $\delta\in(0,1)$ then it is a uniformly continuous map

$$[0,T]\longrightarrow C^{\delta'}(\bar\Omega)$$

for any $\delta'<\delta$.

Pf (i) We need to show that

$$|\varphi|_{C^{(1-\lambda)\varepsilon}([0,T];\,C^{\lambda\delta}(\bar\Omega))} \qquad (*)$$

$$= \sup_t |\varphi(\cdot,t)|_{C^{\lambda\delta}(\bar\Omega)} + \sup_{t\neq s}\frac{|\varphi(\cdot,t)-\varphi(\cdot,s)|_{C^{\lambda\delta}}}{|t-s|^{(1-\lambda)\varepsilon}} < \infty.$$

What we know is that

$$[\varphi]_{C^{\delta,\varepsilon}(\bar Q_T)} = \sup_t\sup_{x\neq y}\frac{|\varphi(x,t)-\varphi(y,t)|}{|x-y|^\delta} + \sup_x\sup_{t\neq s}\frac{|\varphi(x,s)-\varphi(x,t)|}{|s-t|^\varepsilon} < \infty$$

$$|\varphi|_{C^0(\bar Q_T)} = \sup_{x,t}|\varphi| < \infty.$$

The first term in $(*)$ is easy to bound:

$$\sup_t|\varphi(\cdot,t)|_{C^{\lambda\delta}} = \sup_t\left(\sup_x|\varphi| + \sup_{\substack{x\neq y\\(1-\lambda)\delta}}\frac{|\varphi(x,t)-\varphi(y,t)|}{|x-y|^{\lambda\delta}}\right)$$

$$\le |\varphi|_{C^0(\bar Q_T)} + (\operatorname{diam}\Omega)\,[\varphi]_{C^{\delta,\varepsilon}(\bar Q_T)}.$$

The next term is a bit more complicated:

$$\sup_{t\neq s}\frac{|\varphi(\cdot,t)-\varphi(\cdot,s)|_{C^{\lambda\delta}}}{|t-s|^{(1-\lambda)\varepsilon}}$$

$$= \sup_{t\neq s}\sup_x\frac{|\varphi(x,t)-\varphi(x,s)|}{|t-s|^{(1-\lambda)\varepsilon}} + \sup_{t\neq s}\sup_{x\neq y}\frac{|\varphi(x,t)-\varphi(x,s)-(\varphi(y,t)-\varphi(y,s))|}{|x-y|^{\lambda\delta}\,|t-s|^{(1-\lambda)\varepsilon}}$$

$$\le T^{\lambda\varepsilon}[\varphi]_{C^{\delta,\varepsilon}} + [\varphi]^{\lambda}_{C^{\delta,\varepsilon}}[\varphi]^{1-\lambda}_{C^{\delta,\varepsilon}}.$$

(ii) Exercise.

Lem 1.3 Let $\delta \in (0,1)$. Then

(i) $\nabla^\perp(-\Delta)^{-1} : C^\delta(\overline{\Omega}) \to C^{1+\delta}(\overline{\Omega})$ is continuous. Moreover

(ii) $\varphi \in C^{\delta,0}(\overline{Q_T}) \Rightarrow u = \nabla^\perp(-\Delta)^{-1}\varphi$ is $C^{1+\delta',0}(\overline{Q_T})$ $\forall \delta' < \delta$.

(iii) similarly $\varphi \in C^{\delta,\varepsilon}(\overline{Q_T}) \Rightarrow u \in C^{1+\delta',\varepsilon'}$ $\forall \delta'<\delta,\ \varepsilon'<\varepsilon$.

Pf (i) We know $(-\Delta)^{-1} : C^\delta(\overline{\Omega}) \to C^{2+\delta}(\overline{\Omega})$ is bdd, and similarly

$$\frac{\partial}{\partial x_i} : C^{2+\delta}(\overline{\Omega}) \to C^{1+\delta}(\overline{\Omega}).$$

(iii) By Lem 1.2,

$$\varphi \in C^{\delta,\varepsilon}(\overline{Q_T}) \Rightarrow \varphi \in C^{(1-\lambda)\varepsilon}([0,T], C^{\lambda\delta}(\overline{\Omega}))$$

for all $\lambda \in (0,1)$. ~~First choose λ so that $(1-\lambda)\varepsilon = \varepsilon'$. Then~~

Thus (WHY?)

$$u = \nabla^\perp(-\Delta)^{-1}\varphi \in C^{(1-\lambda)\varepsilon}([0,T], C^{1+\lambda\delta}(\overline{\Omega}))$$

First choose $\lambda \in (0,1)$ so that $(1-\lambda)\varepsilon = \varepsilon'$. Then have

$$u \in C^{\varepsilon'}([0,T], C^{1+\lambda\delta}(\overline{\Omega}))$$
$$\Rightarrow D_x u \in C^{\varepsilon'}([0,T], C^{\lambda\delta}(\overline{\Omega}))$$
$$\Rightarrow u \in C^{0,\varepsilon'}(\overline{Q_T}).$$

Similarly with $\lambda \in (0,1)$ so that $\lambda\delta = \delta'$ we get

$$D_x u \in C^{(1-\lambda)\varepsilon}([0,T], C^{\delta'}(\overline{\Omega}))$$
$$\Rightarrow u \in C^{\delta',0}(\overline{Q_T}).$$

Thus $u \in C^{\delta',\varepsilon'}(\overline{Q_T})$ as desired.

(ii) Exercise.

OOPS, COLUMNS REVERSED

What if we just have $\varphi \in L^\infty$? Then we have to directly use the Greens representation

$$u(x) = \int \nabla^\perp_x G^\Omega(x,y)\varphi(y)\,dy.$$

Now from

$$G^\Omega(x,y) = -\frac{1}{2\pi}\log|x-y| + \text{smooth}$$

we get

$$|D_x G^\Omega(x,y)| \le \frac{K}{|x-y|}$$
$$|D_x^2 G^\Omega(x,y)| \le \frac{K}{|x-y|^2}$$

for some constant K.

Lem 1.4 If $\varphi \in L^\infty(\Omega)$, then $(-\Delta)^{-1}\varphi \in C^1(\overline{\Omega})$ and $u = -\nabla^\perp(-\Delta)^{-1}\varphi$ is a tangential flow with

$$|u|_{L^\infty} \le K|\varphi|_{L^\infty}$$
$$|u(x)-u(y)| \le K|\varphi|_{L^\infty}|x-y|\,\chi(|x-y|)$$

where $K = K(\Omega)$ and

$$\chi(s) = (1+\log(1/s))\,1_{0<s<1}$$

PROOF NEXT TIME