# DAY 09

## Epilogue for Leray solns

We showed, roughly (force $f \equiv 0$)

**Leray Thm** For any initial data $u_0 \in H^1_0$, (NS) has a ~~soln~~ weak soln in
$$C_t H^{-1}_x \cap L^\infty_t L^2_x \cap L^2_t H^1_x$$
which satisfies the energy inequality

Q What about uniqueness, continuous dependence on initial data?

**Thm 3.2** In $d=2$ dimensions, Leray solutions are unique. Moreover if we have two solns $u, v$ then we can estimate
$$|u-v|_{L^\infty L^2},\ |u-v|_{L^2 H^1}$$
in terms of $|u_0 - v_0|_{L^2}$ for finite time intervals.

~~Key estimate~~

**Thm 3.3** In $d=3$ dimensions Leray solutions that are ~~in~~ $L^4_t H^1_x$ are unique.

**Thm 3.4** If $u_0 \in H^{1/2}_0$ then $\exists T > 0$ and a unique solution in $L^4([0,T], H^1)$. Moreover, $\exists c > 0$ s.t. $|u_0|_{H^{1/2}} \leq c\nu \Rightarrow T = +\infty$.



## New topic: Navier Stokes vs Euler

- Nondimensionalization
- Paradoxes
- Singular perturbations
- Prandtl boundary layers

Consider the incompressible, constant-density NS eqns:
$$(NS)\begin{cases} \rho(u_t + (u\cdot\nabla)u) = -\nabla p + \mu\Delta u \\ \nabla\cdot u = 0 \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$
where $\mu > 0$ is the dynamic viscosity, which we will always assume is constant. We want to be able to say that $\mu$ is "big" or "small", but right now that's a meaningless statement because the value of $\mu$ depends on our choice of units (m, s, kg, ...).

## Nondimensionalization

The solution to this problem is to introduce

- a velocity scale $U$
- a length scale $L$
- a time scale $T$

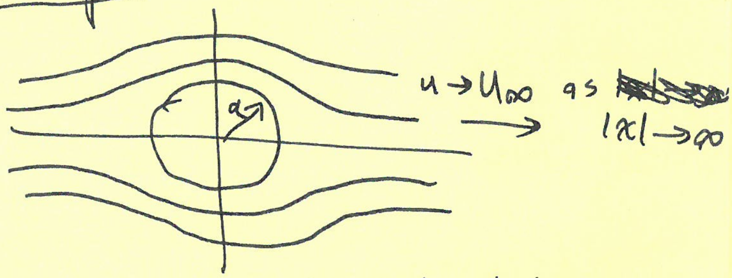
and work with the dimensionless variables
$$x' = \frac{x}{L}, \quad u' = \frac{u}{U}, \quad t' = \frac{t}{T}$$

Of course we can pick any $L, U, T$ we want, but for the following arguments to be justified what we want is for things like

$u', \nabla_{x'} u', \partial_{t'} u'$

to be "$O(1)$ quantities".

Example




seems reasonable to take

$$U = U_\infty$$
$$L = a$$

What about $T$? Usually one just set $T = L/U$. This has the correct units, and is reasonable if we think about the timescales involved in the motion of fluid particles.

Changing variables

$$\begin{cases} \nabla = \nabla_x = \frac{1}{L}\nabla_{x'} =: \frac{1}{L}\nabla' \\ \partial_t = \frac{1}{T}\partial_{t'} = \frac{U}{L}\partial_{t'} \\ u = U u' \\ p =: \rho U^2 p' \end{cases}$$

We get

$$\partial_t u = \frac{U^2}{L}\partial_{t'} u'$$

$$(u\cdot\nabla)u = \frac{U^2}{L}(u'\cdot\nabla')u'$$

$$\nabla p = \rho\frac{U^2}{L}\nabla' p'$$

$$\Delta u = \frac{U}{L^2}\Delta' u'$$

$$\nabla\cdot u = \frac{U}{L}\nabla'\cdot u'$$



and so the momentum eqn becomes,

$$\rho\frac{U^2}{L}(\partial_{t'}u' + (u'\cdot\nabla')u')$$

$$= \rho\frac{U^2}{L}\nabla' p' + \frac{\mu U}{L^2}\Delta' u'$$

Cancelling the common factor of $\rho\frac{U^2}{L}$, we are left with

$$\boxed{\partial_t u' + (u'\cdot\nabla')u' = \nabla' p' + \frac{1}{R}\Delta' u'}$$

where the dimensionless parameter $R$, called the Reynolds number (often written Re) is

$$\boxed{R = \frac{\rho L U}{\mu}}$$

Remark This sort of "similarity" argument is extremely useful, e.g. in experiments. Instead of, say, building a full-size model of an aircraft wing to test in a wind tunnel, we can build a smaller scale model. As long as we adjust e.g. the wind velocity in the tunnel so that the Reynolds #'s in the two configurations are the same, we can expect the results of the small-scale experiment to be relevant for the large-scale one.

Recall that we can ~~absorb~~ eliminate $\nabla p$ from (NS) as well as $\nabla \cdot u = 0$ by using the Leray projection $\mathbb{P}$ onto divergence free vector fields. Get

$$\boxed{\partial_{t'} u' = \mathbb{P}\left(-(u' \cdot \nabla')u' + \frac{1}{R}\Delta u'\right)}$$

As $R \to 0$ we get the <u>Stokes eqns</u>

$$\partial_{t'} u' = \frac{1}{R}\mathbb{P}\Delta u'$$

This is a <u>linear</u> parabolic-type equation, and there is a good theory for it. See, e.g., Chemin's book, Part II Ch. 3.

As $R \to \infty$, on the other hand, we expect to get the Euler eqns

(E) $\partial_{t'} u' = \mathbb{P}(-(u' \cdot \nabla')u')$

This is a <u>singular</u> limit in that

1. The neglected term $\frac{1}{R}\Delta' u'$ was the one with the <u>most</u> derivatives.

2. We usually consider (E) with the <u>slip</u> BC $u \cdot n = 0$ on $\partial\Omega$, but (NS) had the stronger <u>no-slip</u> BC $u = 0$ on $\partial\Omega$.

FROM NOW ON WE <u>DROP THE PRIMES</u>. WE WILL NEED THEM FOR ADDITIONAL SCALINGS.

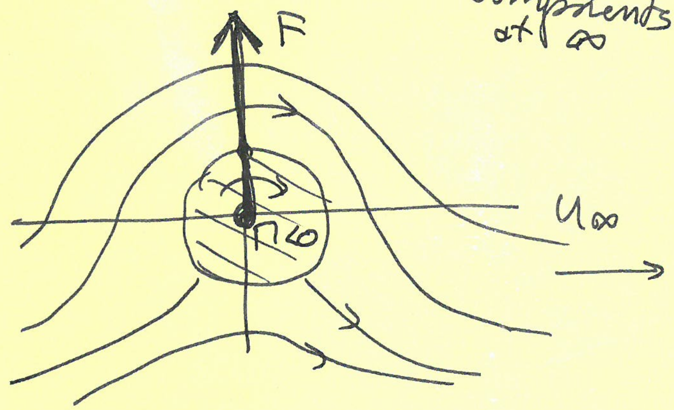


On the other hand for many applications $Re \gg 1$ (Wikipedia: a person swimming has $Re \sim 10^6$) and so we would really like to have a good theory for the $Re \to \infty$ limit.

Just using Euler & slip BC, though, sometimes leads to paradoxical results. For example, we proved the <u>Kutta–Jukowski thm</u> which said that the force on a body in 2d steady potential flow was

$$F = \rho\Gamma(V_\infty - iU_\infty)$$

circulation — velocity components at $\infty$




In particular there is only a "lift" force perpendicular to $(U_\infty, V_\infty)$ and no "drag" force parallel to it.

This seems wrong...

In 3D things are even weirder:

**D'Alembert's paradox**
In steady potential flow around an obstacle in 3D with constant velocity at $\infty$, there is no force of any kind exerted on the body.

How to resolve this? One explanation would be that we need to include vorticity.
But there is a common objection to this: how was the vorticity created? If we imagine that the body and fluid are initially at rest then $u_0 = u|_{t=0} = \text{const.}$ and $\omega_0 = \omega|_{t=0} = \text{const.}$ and so the vorticity eqn

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) u \quad (+ \nu \Delta \omega)$$

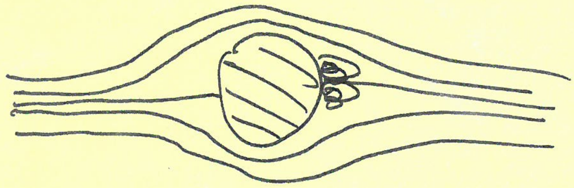
would seem to imply $\omega \equiv 0$ for all time. The explanation is that vorticity is generated at the boundaries, and that this is a "singular" effect of even small viscosity (even large Re).



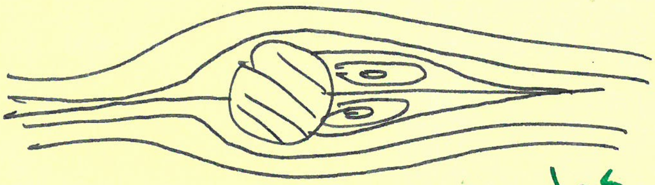
## Boundary Layers

- NS flow is drastically different from Euler in a boundary layer of thickness $1/\sqrt{R}$
- Over time this layer may separate from the boundary
- This is a mechanism for generating vorticity (and hence resolving the D'Alembert paradox).

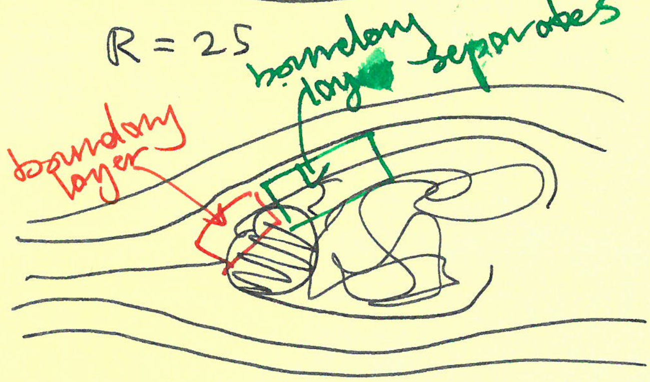
Sketches of beautiful pictures from Von Dyke's "Album of fluid motion"



R = 10



R = 25




R = 2000

## Singular perturbations in ODEs

Consider the problem

$$(P)\quad \begin{cases} \varepsilon y'' + y' = a \\ y(0) = 0 \\ y(1) = 1 \end{cases}$$

where $a \in (0,1)$ is a fixed constant and $0 < \varepsilon \ll 1$ is a small parameter.

[Explicit solution:

$$y = \frac{1-a}{1-e^{-1/\varepsilon}}\left(1 - e^{-x/\varepsilon}\right) + ax$$
]

How can we approximate the soln of (P) when $\varepsilon \ll 1$? Setting $\varepsilon = 0$ we are tempted to look at

$$y' = a$$

but then we can only impose one of the two boundary conditions. Let's choose the one at $x = 1$. Then our "outer problem" is

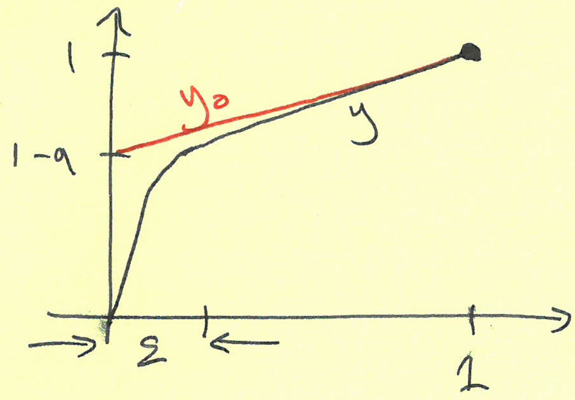
$$(P_0)\quad \begin{cases} y_0' = a \\ y_0(1) = 1 \end{cases}$$

which has the explicit solution

$$y_0 = a(x-1) + 1.$$

But of course $y_0$ does not satisfy the BC at $x=0$ at all. How to fix?






Well, obviously we need some other approximation near $x = 0$. Looking at the eqn (or cheating and looking at the explicit solution) we guess that the "length scale" involved is $\sim \varepsilon$, i.e. that we should work with the dimensionless variable

$$X = \frac{x}{\varepsilon}.$$

Let

$$Y(X) = y(x)$$

and plug into (P). We get

$$\frac{d}{dx} = \frac{1}{\varepsilon}\frac{d}{dX}$$

and so

$$\frac{1}{\varepsilon}\left(\frac{d}{dX}\right)^2 Y + \frac{1}{\varepsilon}\frac{dY}{dX} = a,$$

i.e.

$$Y'' + Y' = \varepsilon a$$

~~Now we're focusing~~

Sending $\varepsilon \to 0$ we just get

$$Y'' + Y' = 0.$$

But what about boundary conditions? The whole point was to get the BC at $x = X = 0$ correct, so

$$Y(0) = 0.$$

For the other BC, we want to "match" with our outer solution, and so we ask for

$$\lim_{X\to\infty} Y(X) = \lim_{x\to 0} y_0(x) = 1-a.$$

This gives us the "inner problem"

$$(P_i)\begin{cases} Y'' + Y' = 0 \\ Y(0) = 0 \\ \lim\limits_{X\to\infty} Y = 1-a \end{cases}$$

which has the soln

$$Y = (1-a)(1-e^{-X})$$

corresponding to

$$y = (1-a)(1-e^{-x/\varepsilon}).$$

Thus our approximation of $y$ is something like

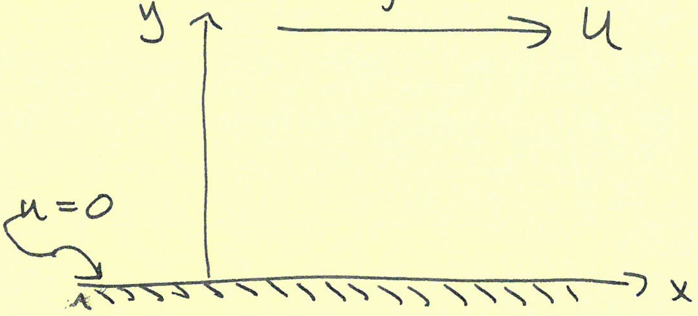
$$y \sim \begin{cases} (1-a)(1-e^{-x/\varepsilon}) & x < \varepsilon \\ a(x-1)+1 & x > \varepsilon \end{cases}$$

(which does a very good job.)



## Boundary layers in NS

Consider a 2D situation with $\vec{u} = u(y,t)e_1$



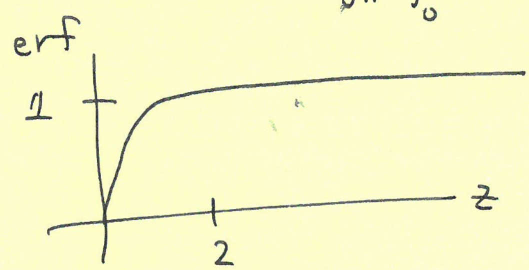

and $u \to U$ as $y \to \infty$. Then (NS) becomes

$$\begin{cases} u_t = \varepsilon u_{yy}, \quad \varepsilon = \frac{1}{R}. \\ u|_{y=0} = 0 \\ \lim\limits_{y\to\infty} u = U \end{cases}$$

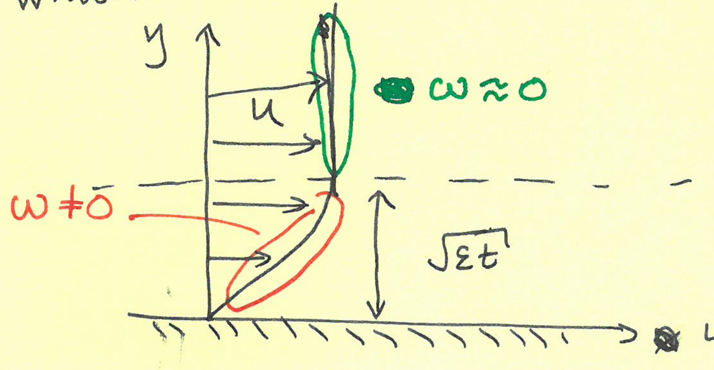
This is just the heat equation, and it has a (scale-invariant) solution

$$u = U \operatorname{erf}\left(\frac{y}{2\sqrt{\varepsilon t}}\right)$$

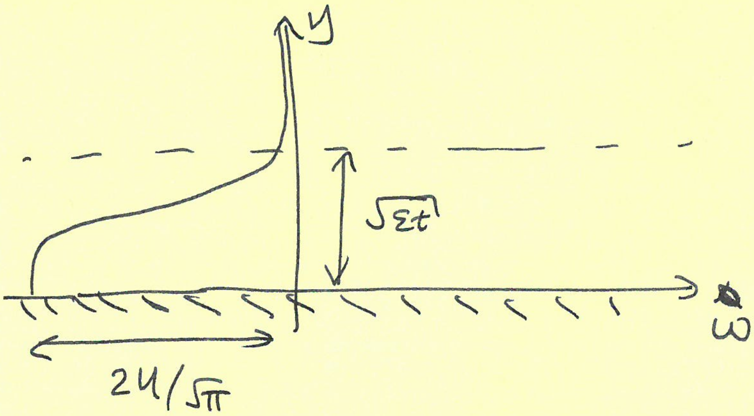
$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2}\, ds$$




which looks like



~~As t increases~~




To understand something more complicated like the separation of a boundary layer, we want to derive approximate equations for its evolution.

Scaling Partially motivated by our explicit example, we use

$$\begin{cases} u' = u \\ x' = x \\ t' = t \\ p' = p \end{cases} \qquad \begin{cases} y' = \dfrac{y}{\delta} \\ v' = \dfrac{v}{\delta} \end{cases}$$

comes from incompress.

where $\delta$ is a small parameter. With $\varepsilon = \frac{1}{R}$, get

$$\begin{cases} u'_{t'} + u'u'_{x'} + v'u'_{y'} = -p'_{x'} + \varepsilon\left(u'_{x'x'} + \frac{1}{\delta^2}u'_{y'y'}\right) \\ \delta\left(v'_{t'} + u'v'_{x'} + v'v'_{y'}\right) = -\frac{1}{\delta}p'_{y'} + \varepsilon\left(\delta v'_{x'x'} + \frac{1}{\delta}v'_{y'y'}\right) \\ u'_{x'} + v'_{y'} = 0 \\ u = v = 0 \quad \text{on } y = 0 \end{cases}$$

Because we want the

$$\frac{\varepsilon}{\delta^2} u'_{y'y'}$$

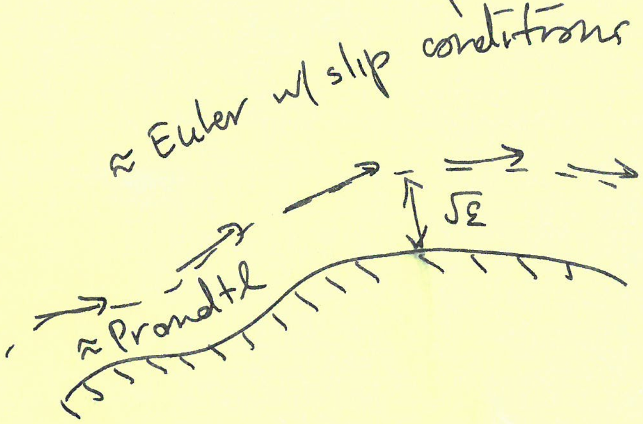
term to balance with the LHS



we set $\delta = \sqrt{\varepsilon}$. Sending $\varepsilon \to 0$ and dropping the primes we get the <u>Prandtl boundary layer eqns</u>:

$$\begin{cases} u_t + uu_x + vu_y = -p_x + u_{yy} \\ p_y = 0 \\ u_x + v_y = 0 \\ u = v = 0 \quad \text{on } y = 0 \end{cases}$$

The hope is that this will be a good "inner problem". In general, we hope to get




... except when there's separation in which case we hope for

The mathematical theory is far from complete!