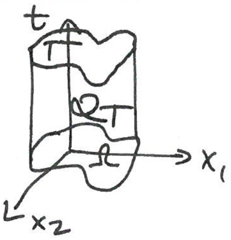
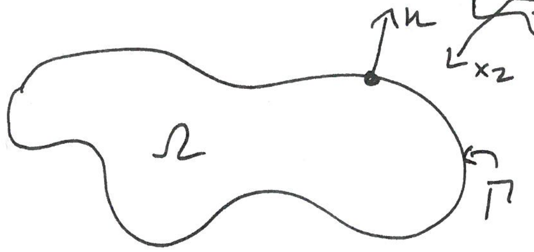
DAY 05





$$
(E)\begin{cases}
\dfrac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla p & \text{in } \Omega \\
\nabla\cdot u = 0 & \text{in } \Omega \\
u\cdot n = 0 & \text{on } \Gamma \\
u|_{t=0} = u_0
\end{cases}
$$

We are proving

Thm (Kato) Suppose $u_0 \in C^{1+\vartheta}(\bar\Omega)$ for some $\vartheta \in (0,1)$, and that

$$\nabla\cdot u_0 = 0 \quad \text{in } \Omega$$
$$u_0\cdot n = 0 \quad \text{on } \Gamma$$

Then $\exists!$ soln (u,p) of (E), with all partial derivatives in (E) being $C(\bar Q_T)$.

p unique up to an additive function of time

General plan

Apply the Schauder fixed-point theorem to the following mapping:

1. Start with a guess $\varphi(x,t)$ for the vorticity.
2. Recover the velocity field $u(x,t)$ via
$$u = -\nabla^\perp(-\Delta)^{-1}\varphi$$
3. Use flow Φ^t coming from u to define
$$\underbrace{\omega(\Phi^t(x),t) = \omega_0(x,t)}$$

For a true soln, have $\frac{D\omega}{Dt} = 0$ and so
$$\omega(\Phi^t,t) = \omega(x,t)$$

Last time we studied

$$\nabla^\perp(-\Delta)^{-1}: C^\delta(\bar\Omega) \to C^{1+\delta}(\bar\Omega)$$
$$C^{\delta,0}(\bar Q_T) \to C^{1+\delta'}(\bar Q_T)$$
$$C^{\delta,\varepsilon}(\bar Q_T) \to C^{1+\delta',\varepsilon'}(\bar Q_T)$$

"# of derivatives in x"

"# of derivatives in t"

Today we start with

$$\nabla^\perp(-\Delta)^{-1}: L^\infty(\Omega) \to C^0(\bar\Omega).$$

Recall that

$$(-\Delta)^{-1}\varphi(x) = \int g(x,y)\varphi(y)\,dy$$

where

$$g(x,y) = -\frac{1}{2\pi}\log|x-y| + \text{smooth terms}.$$

In particular,

$$|D_x g(x,y)| \le \frac{K}{|x-y|},$$
$$|D_x^2 g(x,y)| \le \frac{K}{|x-y|^2}$$

for some constant $K = K(\Omega)$.

Inverting the curl

Lem 1.4 If $\varphi \in L^\infty(\Omega)$, then $(-\Delta)^{-1}\varphi \in C^1$ and $u = -\nabla^\perp(-\Delta)^{-1}\varphi$ is a tangential flow with

$$(*) \quad |u|_{L^\infty} \le K|\varphi|_{L^\infty},$$
$$(**) \quad |u(x)-u(y)| \le K|\varphi|_{L^\infty}|x-y|\,\chi(|x-y|)$$

where

$$\chi(s) = (1+\log(1/s))\,1_{0<s<1}$$

☆ Somehow this is the key estimate in the paper.

Pf ~~[scribble]~~ Skipping some technicalities, we have

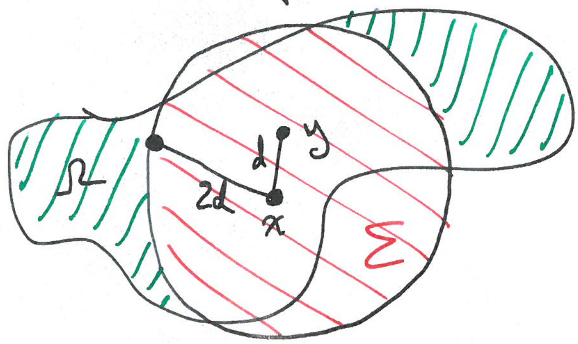
$$u(x) = -\nabla_x^\perp (-\Delta)^{-1}\varphi$$
$$= -\nabla_x^\perp \int g(x,y)\varphi(y)\,dy$$
$$= -\int \nabla_x^\perp g(x,y)\varphi(y)\,dy$$

and hence

$$|u(x)| \le \int_\Omega \frac{K}{|x-y|}|\varphi(y)|\,dy$$
$$\le K|\varphi|_{L^\infty} \int_\Omega \frac{dy}{|x-y|} \longrightarrow \int \frac{r\,dr}{r}$$
$$\le \tilde K |\varphi|_{L^\infty},$$

which is (*).

Next to prove (**) let



Then

$$|u(x)-u(y)| \le |\varphi|_{L^\infty} \int_{\Omega\cap\Sigma} \left(|\nabla_x^\perp g(x,z)| + |\nabla_y^\perp g(y,z)|\right) dz$$
$$+ |\varphi|_{L^\infty} \int_{\Omega\setminus\Sigma} |\nabla_x^\perp g(x,z) - \nabla_y^\perp g(y,z)|\,dz$$
$$= \mathrm{I} + \mathrm{II}$$

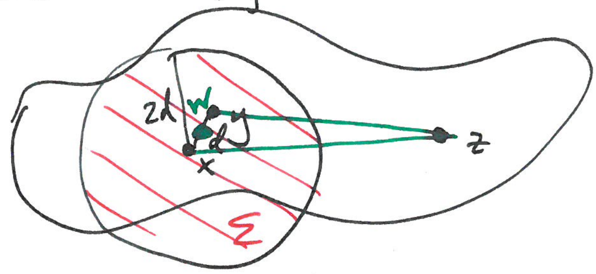
As before

$$\mathrm{I} \le K|\varphi|_{L^\infty}\left(\int_\Sigma \frac{dz}{|x-z|} + \int_\Sigma \frac{dz}{|y-z|}\right)$$
$$\le 10\pi K d\, |\varphi|_{L^\infty}.$$

For II, we estimate

$$|\nabla_x^\perp g - \nabla_y^\perp g| = |x-y|\,|D_w^2 g(w,z)|$$
$$\le \frac{Kd}{|w-z|^2} \le \frac{4Kd}{|x-z|^2}$$

where "$w \in [x,y]$".



Thus

$$\mathrm{II} \le |\varphi|_{L^\infty} 4Kd \int_{2d\le|x-z|\le R} \frac{dz}{|x-z|^2}\,dz \qquad R = \operatorname{diam}\Omega$$

$$= 8\pi K d \log\left(\frac{R}{2d}\right)|\varphi|_{L^\infty}.$$

~~Easy to check~~

Since $(-\Delta)^{-1}\varphi = 0$ on Γ,

$$n\cdot u = -n\cdot\nabla^\perp[(-\Delta)^{-1}\varphi]$$
$$= \pm\tau\cdot\nabla[(-\Delta)^{-1}\varphi]$$
$$= 0$$

there. Can also check that $\nabla\cdot u = 0$ in the sense of distributions □

Now we would expect

$$(-\Delta)^{-1}\nabla\cdot$$

to have similar properties to

$$\nabla^\perp(-\Delta)^{-1},$$

but there is a wrinkle that for $f \in C^s(\bar\Omega, \mathbb{R}^2)$ we must ~~integr~~ interpret $\nabla\cdot f$ as a distribution. ~~But this all works out, and we get~~

If $f \in C^1(\bar\Omega)$, then

$$(-\Delta)^{-1}(\nabla\cdot f) = \int_\Omega g(x,y)\, \nabla\cdot f(y)\, dy$$

$$= -\int_\Omega \nabla_y g(x,y)\cdot f(y)\,dy,$$

~~and~~ The RHS makes sense for ~~any~~ $f \in C^0(\bar\Omega)$, and so we <u>define</u>

$$(-\Delta)^{-1}\nabla\cdot f = -\int_\Omega \nabla_y g(x,y)\cdot f(y)\,dy$$

for $f\in C^0(\bar\Omega)$. Easy to check that $(-\Delta)^{-1}\nabla\cdot : C^0(\bar\Omega) \to C^0(\bar\Omega)$, and that $\psi = (-\Delta)^{-1}\nabla\cdot f$ is a weak soln to $-\Delta\psi = \nabla\cdot f$ in that $\psi|_\Gamma = 0$ and

$$(\psi, \Delta\Phi) = (f, \nabla\Phi)$$

for all $\Phi \in C^2(\bar\Omega)$ with $\Phi|_\Gamma = 0$.

<u>Lem 1.5</u> For $\delta\in(0,1)$, $(-\Delta)^{-1}\nabla\cdot : C^\delta(\bar\Omega) \to C^1(\bar\Omega)$ is continuous. If $f \in C^{\delta,0}(\bar Q_T)$, then $(-\Delta)^{-1}\nabla\cdot f \in C^{1,0}(\bar Q_T)$.

<u>Pf</u> See Kato.

<u>Lem 1.6</u> Suppose that $w \in C(\bar\Omega, \mathbb{R}^2)$ satisfies $(w, \nabla^\perp\Phi) = 0$ $\forall \Phi\in C_0^\infty(\Omega)$. Then $w = \nabla p$ for some $p \in C^1(\bar\Omega)$. If $w\in C(\bar Q_T)$ then we can choose $p \in C^{1,0}(\bar Q_T)$.

<u>Pf</u> See Kato. [If things were smooth, then IBP $\Rightarrow$ $\nabla^\perp\cdot w = 0$ $\Rightarrow$ $w = \nabla p$]

§2 Construction of the solution

We want to apply the Schauder fixed-point theorem, and for this we need a Banach space X and a convex compact subset $S \subseteq X$. We will use $X = C^0(\bar Q_T)$; S will be a proper subset of

$$S' = \bigcup_{\varepsilon\in(0,1)} C^{\varepsilon,0}(\bar Q_T)$$

to be determined.

"Inverting curl"

<u>Lem 2.1</u> For $\varphi \in S'$,

$$u := \nabla^\perp(-\Delta)^{-1}\varphi \in C^{1,0}(\bar Q_T)$$

satisfies

$$-\nabla^\perp\cdot u = \varphi.$$

<u>Pf</u> Since $\varphi \in C^{\varepsilon,0}$ for some $\varepsilon > 0$, Lemma 1.3 implies that $u \in C^{1+\varepsilon',0} \subseteq C^{1,0}$. Moreover

$$-\nabla^\perp\cdot u = -\nabla^\perp\cdot\nabla^\perp(-\Delta)^{-1}\varphi = (-\Delta)(-\Delta)^{-1}\varphi = \varphi.$$

□

Now we want to construct the flow map Φ^t.

Note that by lem 1.4, u is a <u>tangential flow</u>!

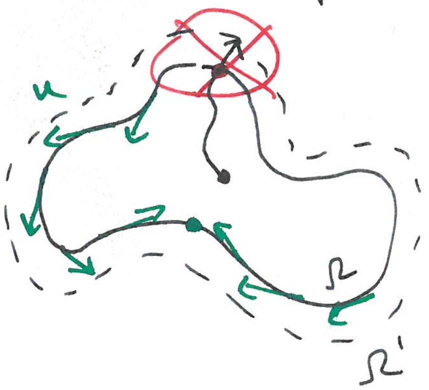
Lem 2.2 Let u be as in Lem 2.2. Then the ODE

$$\dot{X} = u(X,t), \quad 0 \le t \le T$$
$$X(0) = y \in \bar{\Omega}$$

has a unique global solution

$$X(t) = \Phi^t(y).$$

Pf Since $u \in C^{1,0}(\bar{Q}_T)$, there is a unique local solution for any $y \in \Omega$. To extend to $0 \le t \le T$, first extend u in a $C^{1,0}$ way to a nbhd Ω' of $\bar{\Omega}$. Then every solution starting at $y \in \bar{\Omega}$ can be continued until it hits $\partial Q_T'$ ($Q_T' = \Omega' \times [0,T]$). So it is enough to prove that X cannot hit $\Gamma \times [0,T]$. Because of local uniqueness, it is enough to show that solutions on Γ stay there. This follows from u being a tangential flow (see Kato for a more involved explanation). □



Lem 2.3 The map $\Phi^t(y)$ from Lem 2.2 is C^1 in t & y. For fixed t it is a one-to-one measure preserving map $\bar{\Omega} \to \bar{\Omega}$, with Jacobian determinant 1. $\Phi^0 = \mathrm{id}$. Similarly for $(\Phi^t)^{-1}$.

Pf Given properties of u, these are standard ODE facts. □

We can finally define our fixed point map F. Given $\varphi \in S'$, let $u = \nabla^\perp(-\Delta)^{-1}\varphi \in C^{1,0}$ as in Lem 2.1. Setting

$$\omega_0 = -\nabla^\perp \cdot u_0 \in C^0(\bar{\Omega}),$$

define $\omega \in C^0(\bar{Q}_T)$ by

$$\omega(x,t) = \omega_0((\Phi^t)^{-1}(x)).$$

Define $F : S' \to C^0(\bar{Q}_T)$ by

$$F : \varphi \longmapsto \omega.$$

If things were smooth, we could check that $\omega = F[\varphi]$ solved

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = 0.$$

But it is at least a weak soln:

Lem 2.4 For $\Psi \in C^1(\bar{\Omega})$,

$$\frac{d}{dt}(\omega, \Psi) = (\omega, u \cdot \nabla \Psi)$$

Pf

$$(\omega, \Psi) = \int \omega_0((\Phi^t)^{-1}(x))\, \Psi(x)\, dx = \int \omega_0(x)\, \Psi(\Phi^t(x))\, dx$$

and so

$$\frac{d}{dt}(\omega, \Psi) = \int \omega_0(x) \frac{\partial}{\partial t} \Psi(\Phi^t(x))\, dx = \int \omega_0(x)\, (u \cdot \nabla \Psi)(\Phi^t(x), t)\, dx$$

$$= \int \omega_0(x)\, (u \cdot$$

$$= \int \omega_0((\Phi^t)^{-1}(x))\,(u\cdot\nabla\Psi)(x,t)\,dx$$
$$= \int \omega\,(u\cdot\nabla\Psi)(x,t)\,dx$$
$$= (\omega, u\cdot\nabla\Psi). \quad \square$$

Lem 2.6 Let $\Phi^t, (\Phi^t)^{-1}$ be as in Lem 2.2. Then

(*) $|\Phi^t(y) - \Phi^{\bar t}(\bar y)| \le C(|y-\bar y|^\delta + |t-\bar t|^\delta)$

where C, δ depend only on $|\varphi|_{L^\infty}$. Similarly for $(\Phi^t)^{-1}$.

Pf Can assume that $|y-\bar y|, |t-\bar t| \le 1$.

Enough to prove (*) in two cases,
(i) $t = \bar t$,
(ii) $y = \bar y$.

For case (i), let $x(t) = \Phi^t(y)$ and $\bar x(t) = \Phi^t(\bar y)$. Then by Lem 1.4,

$$\left|\frac{d}{dt}(x(t)-\bar x(t))\right| = |u(x(t),t) - u(\bar x(t),t)|$$
$$\le C|x(t)-\bar x(t)|\,\chi(|x(t)-\bar x(t)|).$$

Thus $\rho(t) = |x(t)-\bar x(t)|$ has

$$|\rho'(t)| \le C\rho(t)(1-\log\rho(t)). \quad (**)$$

Playing around with (**), one can show that

$$|y-\bar y| < e^{-e^{LT}} \Rightarrow$$
$$|\Phi^t(y) - \Phi^t(\bar y)| \le e|y-\bar y|^{e^{-LT}} < 1.$$

[Rk In addition to being good for $(-\Delta)^{-1}$, Hölder spaces are very well-suited to problems involving flow maps Φ^t.]

For (ii), we have

$$|x(t) - x(\bar t)| = \left|\int_{\bar t}^t u(x(s),s)\,ds\right|$$
$$\le |u|_{L^\infty}|t-\bar t|$$
$$\text{(Lem 1.4)} \quad \le K|\varphi|_{L^\infty}|t-\bar t|. \quad \square$$

Putting this all together, we get

Lem 2.7 $\omega = F(\varphi)$ satisfies

$$|\omega|_{L^\infty} \le |\omega_0|_{L^\infty}$$
$$|\omega(x,t) - \omega(y,t)| \le C(|x-y|^\delta + |t-s|^\delta)$$

where C, δ depend only on $|\varphi|_{L^\infty}$.

Pf Use Lem 2.6 and $\omega(x,t) = \omega_0((\Phi^t)^{-1}(x))$. $\square$

Def Let S be the set of all $\varphi(x,t)$ such that $|\varphi|_{L^\infty} \le M$ and $|\varphi(x,t) - \varphi(y,s)| \le C(|x-y|^\delta + |t-s|^\delta)$ where C, δ are the constants from Lem 2.7 (when $|\varphi|_{L^\infty} = M$).

Clearly $S \subseteq S'$, and Lem 2.7 shows that $F: S \to S$. Also S is clearly a convex and compact subset of $C^0(\bar Q_T)$.

$\Delta \neq$ A-A

~~Lem 2.8 $F: C^0(\bar Q_T) \to C^0(\bar Q_T)$ is continuous.~~

Lem 2.8 F is continuous in the $C^0(\bar Q_T)$ topology.

Pf Let $\varphi, \varphi^{(1)}, \varphi^{(2)}, \ldots \in S$ and suppose $|\varphi^{(n)} - \varphi|_{L^\infty} \to 0$. Defining $u^{(n)}, \omega^{(n)}, u, \omega$ in the obvious way, easy to check that

$$|u^{(n)} - u|_{L^\infty} = |\nabla^\perp(-\Delta)^{-1}(\varphi^{(n)} - \varphi)|_{L^\infty}$$

$$\text{(Lem 1.4)} \quad \le k|\varphi^{(n)} - \varphi|_{L^\infty} \to 0.$$

Theorem from ODEs gives $\Phi^t_{(n)} \to \Phi^t$, uniformly in x & t, and hence $|\omega^{(n)} - \omega|_{L^\infty} \to 0$. □

$\omega(x,t) = \omega_0((\Phi^t)^{-1}(x)) \in C^\theta$

So by the Schauder fixed-point theorem, F has at least one fixed point $\varphi \in S$.

§3 Existence of a solution

Let $\varphi \in S$ be the fixed-point from §2, and let Φ^t, u be as usual.

Lem 3.1 $D_x u \in C^{\theta', \theta'}(\bar{Q}_T)$ for any $\theta' < \theta$.

Pf Since $\omega_0 \in C^\theta(\bar{\Omega})$, and $\Phi \in C^1(\bar{Q}_T)$ (Lem 2.3), easy to check that $\varphi \in C^{\theta,\theta}$. Then apply Lem 1.3 to $u = \nabla^\perp(-\Delta)^{-1}\varphi$. □

Lem 3.2 $D_t u \in C^0(\bar{Q}_T)$.

Pf Testing against $\Psi \in C^2(\bar{\Omega}, \mathbb{R}^2)$, can check that

$$\frac{d}{dt}(u, \Psi) = (\nabla^\perp(-\Delta)^{-1}\nabla\cdot(\varphi u), \Psi)$$

Since by Lem 1.5

$$\underline{\nabla^\perp(-\Delta)^{-1}\nabla\cdot(\varphi u)} \in C(\bar{Q}_T),$$

we conclude that

$$\frac{\partial u}{\partial t} = \qquad \in C(\bar{Q}_T). \quad \square$$

Lem 3.3 $\exists\, p \in C^{1,0}(\bar{Q}_T)$ so that (u,p) solves (E).

Pf Use Lemmas 1.6 & 2.4; see Kato. □

It remains to show that the solution is unique. Suppose that (u_1, p_1) and (u_2, p_2) are solutions and set

$$v = u_1 - u_2,$$
$$q = p_1 - p_2.$$

Subtracting (E_1) and (E_2) we obtain

$$(***) \quad \frac{\partial v}{\partial t} \pm (u_1\cdot\nabla)v \pm (v\cdot\nabla)u_2 = -\nabla q.$$

Testing against v there are some cancellations and we get

$$\frac{1}{2}\frac{d}{dt}|v|^2_{L^2} \pm (v, (v\cdot\nabla)u_2) = 0.$$

Since $u_2 \in C^1(\bar{Q}_T)$, this implies

$$\frac{d}{dt}|v|^2_{L^2} \le C|v|^2_{L^2}.$$

Since $v|_{t=0} = 0$, we conclude that $v \equiv 0$. Then $(***)$ gives $\nabla q \equiv 0$, and so q is a function of time t only.

END OF PROOF OF THEOREM!