NON-SYMMETRIC SOLUTIONS TO AN OVERDETERMINED PROBLEM FOR THE HELMHOLTZ EQUATION IN THE PLANE

MILES H. WHEELER

Abstract. In this note we construct smooth bounded domains $\Omega \subset \mathbb{R}^2$, other than disks, for which the overdetermined problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = b & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial n} = c & \text{on } \partial\Omega \end{cases}$$

has a solution for some constants $\lambda, b, c \neq 0$. These appear to be the first counterexamples to a conjecture of Willms and Gladwell [WG94].

1. Introduction

In this introduction we consider overdetermined elliptic problems of the form

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = b \qquad \text{on } \partial\Omega, \tag{1.1b}$$

$$\frac{\partial u}{\partial n} = c \qquad \text{on } \partial\Omega, \tag{1.1c}$$

where λ, b, c are constants, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, and the function u is non-constant. There are explicit radially symmetric solutions to such problems when Ω is a ball, and it is natural to ask whether these are the only solutions.

The conjecture that balls are the only solutions of (1.1) when $c = 0$, attributed to Schiffer [Yau82, Problem 80], is a famous and long-standing open problem in spectral geometry. Its interest is due in part to its connection to the Pompeiu problem; see [BST73, Wil76, Ber80] and the survey [Zal92]. While the Schiffer conjecture remains open, there are many partial results showing that it holds under additional assumptions, including [Ber80, BY87, Avi86, Den12, KL20, Mon23]. In particular, Agranovsky [Agr93] (in two dimensions) and Kobayashi [Kob93] (in any dimension) independently showed that the conjecture holds for small perturbations of balls, and so there is no hope of finding counterexamples using local bifurcation arguments. Also see [Can14] for a related result which only requires (1.1c) to hold in an average sense. In terms of negative results, Shklover [Shk00] showed that a generalization of Schiffer's conjecture to Riemannian manifolds is false, and recently Fall, Minlend, and Weth [FMW24] found counterexamples on the sphere $\mathbb{S}^2$ – disproving a conjecture of Souam [Sou05] – as well as a family of non-trivial solutions bifurcating from cylinders when Ω is allowed to be unbounded. Even more recently, Enciso, Fernández, Ruiz, and Sicbaldi [EFRS25] have constructed solutions bifurcating from two-dimensional annuli with different values of the constant b on each boundary component.

The analogue of Schiffer's conjecture for $b = 0$ – so that λ is a Dirichlet rather than Neumann eigenvalue – was posed by Berenstein [Ber80] and is also still open. Although less well-studied than the Neumann version, there are still some partial results. For instance, the works [Ber80, BY87] mentioned above contain related results on both conjectures. As in the Neumann case, there are also counterexamples to generalized versions of the conjecture. Shklover [Shk00] constructed examples on certain Riemannian manifolds, and Sicbaldi found positive solutions involving unbounded domains bifurcating from cylinders [Sic10]; also see [SS12]. More recently, Minlend [Min23] and Dai and Zhang [DZ23] have found sign-changing solutions involving unbounded domains.

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Several authors mention an analogue of Schiffer's conjecture without any restrictions on the constants b, c, and to the best of our knowledge this question has also remained open. The earliest statement of the conjecture appears to be by Willms and Gladwell [WG94], who proved that it holds under the assumption that u has no saddle points. In a subsequent paper with Chamberland [WCG95] they also discussed 'dual' formulations of the problem. Williams [Wil02] stated a more precise version of the conjecture and studied (among other things) the analyticity of solutions, while Souam [Sou05] proved an analogue of the conjecture on the sphere $\mathbb{S}^2$ when $\lambda = 2$. Dalmasso [Dal10] showed that, in two dimensions and under some assumptions on the domain Ω, for any fixed $c \neq 0$ there can be at most finitely many pairs (λ, b) for which (1.1) has a solution. In a later paper, Dalmasso [Dal14] proved that the conjecture holds if either λ is at most the first Dirichlet eigenvalue of the Laplacian, or if Ω is convex and symmetric about a hyperplane and λ is at most the second Dirichlet eigenvalue. While (1.1) with both $b, c \neq 0$ is no longer an overdetermined eigenvalue problem for the Laplacian, after making the simple transformation $u \mapsto u + b$ it can be thought of as an overdetermined semilinear eigenvalue problem

$$\Delta u + \lambda f(u) = 0 \ \text{ in } \Omega, \qquad u = 0, \ \frac{\partial u}{\partial n} = c \ \text{ on } \partial\Omega, \tag{1.2}$$

with the specific nonlinearity $f(u) = u + b$. This form of the problem was studied by Canuto and Rial [CR08], who showed that the unit ball is an isolated solution provided $\lambda > 0$ lies outside a certain countable set. Canuto [Can11] later proved a related result with a less restrictive hypothesis on λ but, at least in the context of (1.2), a more restrictive hypothesis on c.

By Serrin's result [Ser71], semilinear problems of the form (1.2) can have a solution u with a strict sign only when Ω is a ball. The existence of sign-changing solutions for bounded domains Ω other than balls, however, has remained open until quite recently. The only result we are aware of is due to Ruiz [Rui25], who constructed sign-changing solutions to (1.2) using nonlinearities of the form $f(u) = u - (u^+)^3$ in dimensions 2, 3, and 4. For sign-changing solutions in unbounded domains, see the references [Min23, DZ23] mentioned earlier. Lastly, we mention the construction by Kamburov and Sciaraffia [KS21] of solutions in annular domains with nonlinearity $f(u) = 1$, where (as in [EFRS25]) they allow the constants in the boundary conditions to differ on the two boundary components.

In this note, we use local bifurcation techniques to construct families of solutions to (1.1) close to the unit disk in two dimensions. The leading-order expansions for these solutions imply that they are not disks, and so the above Willms–Gladwell conjecture is false, at least in two dimensions. As a construction of sign-changing solutions to an overdetermined semilinear problem (1.2), our proof is much simpler than that of Ruiz [Rui25], which is to be expected given our much simpler nonlinearity and our restriction to two dimensions. Also, unlike in [Rui25] our functions u are real-analytic.

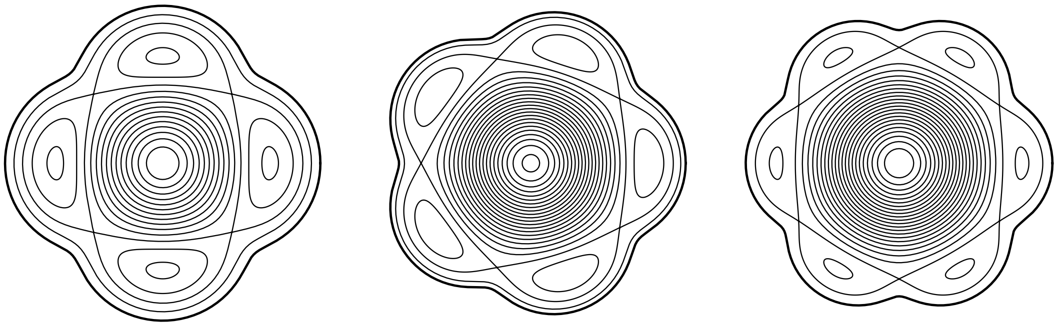
1.1. **Statement of the main result.** Before stating our result more precisely, we need the following lemma about Wronskians of Bessel functions. Here and in what follows we use the standard notation J_ν for the Bessel function of the first kind with order ν.

Lemma 1.1. *For any integer $m \geq 4$, the Wronskian $\mathcal{W}_{1,m} := J_1 J_m' - J_m J_1'$ has a smallest positive root $\mu_m > 0$. Moreover, this root μ_m is simple, strictly decreases as a function of m, and satisfies the inequalities $j_{1,1} < \mu_m < j_{0,2}$, where here $j_{1,1} \approx 3.8317$ is the first positive root of J_1 and $j_{0,2} \approx 5.5201$ is the second positive root of J_0.*

Theorem 1.2. *Fix $n = 2$, $b = 1$, and $\alpha \in (0, 1)$. For any integer $m \geq 4$, there exists $\varepsilon_0 > 0$ and a curve of classical solutions to* (1.1)*, parametrized by $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, with the following properties.*

(i) *For $\varepsilon \neq 0$, the domain $\Omega(\varepsilon)$ is not a disk.*
(ii) *The solutions are m-fold symmetric, in the sense that $\Omega(\varepsilon), u(\varepsilon)$ are invariant under rotations by an angle $2\pi/m$ as well as reflections across the horizontal axis.*
(iii) *The domains $\Omega(\varepsilon)$ are described by conformal mappings $\phi(\varepsilon) \colon \mathbb{D} \to \Omega(\varepsilon)$, where $\mathbb{D}$ is the unit disk, and the unknowns*

$$(u \circ \phi, \phi, c, \lambda) \in C^{2+\alpha}(\overline{\mathbb{D}}) \times C^{2+\alpha}(\overline{\mathbb{D}}, \mathbb{C}) \times \mathbb{R}^2 \tag{1.3}$$



$m = 4$ $m = 5$ $m = 6$

FIGURE 1. Exaggerated sketches of the domains $\Omega(\varepsilon)$ and functions $u(\varepsilon)$ from Theorem 1.2 for $m = 4, 5, 6$, based on the leading-order approximation (1.4).

depend real-analytically on ε.

(iv) *As* $\varepsilon \to 0$ *we have the asymptotic expansions*

$$\phi(re^{i\theta};\varepsilon) = re^{i\theta} + \varepsilon(re^{i\theta})^{m+1} + O(\varepsilon^2), \tag{1.4a}$$

$$(u\circ\phi)(re^{i\theta};\varepsilon) = \frac{J_0(\mu_m r)}{J_0(\mu_m)} + \varepsilon\mu_m\left(\frac{J_1(\mu_m)J_m(\mu_m r)}{J_0(\mu_m)J_m(\mu_m)} - \frac{J_1(\mu_m r)}{J_0(\mu_m)}r^{m+1}\right)\cos m\theta + O(\varepsilon^2), \tag{1.4b}$$

$$c(\varepsilon) = -\mu_m\frac{J_1(\mu_m)}{J_0(\mu_m)} + O(\varepsilon^2), \tag{1.4c}$$

$$\lambda(\varepsilon) = \mu_m^2 + O(\varepsilon^2) \tag{1.4d}$$

in the spaces (1.3).

Remark 1.3. For each fixed ε, the domains $\Omega(\varepsilon)$ are in fact real-analytic. As observed by Williams [Wil02], this follows from [KN77, Theorem 2]. The functions $u(\,\cdot\,;\varepsilon)$ are therefore also real-analytic up to the boundary.

Remark 1.4. Possibly after shrinking $\varepsilon_0 > 0$, the first conclusion (i) of Theorem 1.2 is an immediate consequence of (1.4a). The asymptotic formulas (1.4) are illustrated in Figure 1 for $m = 4, 5, 6$, using an unreasonably large value of $\varepsilon > 0$ so that the saddles and local extrema of u are more apparent. Note that the existence of at least one saddle point is guaranteed by [WG94, Theorem 1]. While the arrangement of critical points in the figure is qualitatively correct, the domains $\Omega(\varepsilon)$ of the exact solutions with small ε will necessarily be convex.

Remark 1.5. If (Ω, u) solves (1.1) with parameters (λ, b, c), then, for any constants $A > 0$ and $B \in \mathbb{R}$, so does $(A^{-1}\Omega, Bu(A\,\cdot\,))$ with parameters $(A^2\lambda, Bb, ABc)$. Thus, provided $b \neq 0$, we are free to set $b = 1$ in Theorem 1.2 without loss of generality. Similarly, we are free to fix one of the other parameters λ, c (provided they are nonzero) or alternatively to fix the scale of Ω. We choose the latter, normalizing the conformal mapping to satisfy $\phi'(0) = 1$; see (2.10) below.

Remark 1.6. As we will see in Section 4, the bounds on μ_m in Lemma 1.1 imply that $J_0(\mu_m)$ and $J_1(\mu_m)$ are negative while $J_m(\mu_m)$ is positive. Thus the denominators appearing in (1.4) are nonzero, the $O(\varepsilon)$ terms in (1.4a) do not vanish identically, and (1.4c) implies $c(\varepsilon) < 0$ for $|\varepsilon|$ sufficiently small. One can also check that, as must be the case, μ_m lies in the discrete set Λ defined in [CR08, Lemma 3.7 and Definition 3.9] where their local uniqueness proof fails.

1.2. **Outline of the paper.** In Section 2, we transform (1.1) into an abstract operator equation $\mathscr{F}(v, w, \gamma; \mu) = 0$ where $\mathscr{F}\colon \mathscr{X} \times I \to \mathscr{Y}$ is a real-analytic mapping between Banach spaces. There are many ways to effect such a transformation; we use conformal mappings combined with a change of dependent variables which partially decouples the linearized equations. In Section 3, we prove Theorem 1.2, assuming that Lemma 1.1 holds, by applying a real-analytic version of the standard Crandall–Rabinowitz theorem [CR71]. In part because of the careful choice of variables in Section 2, the required analysis of the linearized operators $\mathscr{L}(\mu)$ quickly boils down to questions about the Wronskian $\mathcal{W}_{1,m}$ appearing in Lemma 1.1, which we then finally prove in Section 4 using ideas from [Pá13].

2. Reformulation

As is usual in bifurcation analyses of free boundary problems, we begin by reducing (1.1) to a problem in a fixed domain by introducing an appropriate diffeomorphism which becomes a new unknown. Since we are in two dimensions, conformal mappings are a natural option which simplify some of the calculations, but this choice is not essential. To streamline the linear analysis, we then make a further change of dependent variables which mixes the unknown u and this conformal mapping.

2.1. Preliminaries.

To avoid dealing with square roots in some of the calculations below, we replace the boundary condition (1.1c) with half of its square,

$$\frac{1}{2}\left(\frac{\partial u}{\partial n}\right)^2 = \frac{1}{2}c^2 \qquad \text{on } \partial\Omega. \tag{2.1}$$

This is of course equivalent provided we choose the sign of c appropriately. For similar reasons, we switch from the parameter $\lambda > 0$ to

$$\mu := \sqrt{\lambda}. \tag{2.2}$$

We will assume throughout that μ lies in the interval

$$I := (j_{1,1}, j_{0,2}), \tag{2.3}$$

where, as in Lemma 1.1, $j_{1,1} \approx 3.8317$ is the first positive root of J_1 and $j_{0,2} \approx 5.5201$ is the second positive root of J_0.

2.2. Conformal change of variables.

Fixing once and for all a Hölder parameter $\alpha \in (0, 1)$, suppose that $w \in C^{2+\alpha}(\overline{\mathbb{D}}, \mathbb{R}^2)$ is a holomorphic vector field on the unit disk $\mathbb{D} \subset \mathbb{R}^2 \cong \mathbb{C}$, i.e. its components (w_1, w_2) satisfy the Cauchy–Riemann equations

$$\partial_1 w_1 - \partial_2 w_2 = 0, \tag{2.4a}$$
$$\partial_2 w_1 + \partial_1 w_2 = 0 \tag{2.4b}$$

in $\mathbb{D}$. Provided $\|w\|_{C^{2+\alpha}(\mathbb{D})}$ is sufficiently small, the near-identity map $\phi = \mathrm{id} + w$ is a diffeomorphism onto its image, which is a $C^{2+\alpha}$ domain Ω. Letting $\tilde{u} = u \circ \phi \in C^{2+\alpha}(\overline{\mathbb{D}})$, it is straightforward to check that the problem (1.1a), (1.1b), (2.1) with $b = 1$ and $\lambda = \mu^2$ is then equivalent to

$$\Delta\tilde{u} + \mu^2 \det(I + Dw)\,\tilde{u} = 0 \qquad \text{in } \mathbb{D}, \tag{2.4c}$$
$$\tilde{u} = 1 \qquad \text{on } \partial\mathbb{D}, \tag{2.4d}$$
$$\tfrac{1}{2}\tilde{u}_r^2 - \tfrac{1}{2}c^2 \det(I + Dw) = 0 \qquad \text{on } \partial\mathbb{D}, \tag{2.4e}$$

where a subscript r is shorthand for an application of the usual radial derivative $\partial_r = |x|^{-1}x \cdot \nabla$. For algebraic convenience we have also multiplied both (1.1a) and (2.1) by the positive function $\det(I + Dw)$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we note that $\det(I + Dw) = |1 + w'|^2$ where w' is the complex derivative of the holomorphic function w.

The transformed problem (2.4) has a family of "trivial" radially symmetric solutions, expressed in polar coordinates (r, θ) as

$$\tilde{u} = U(r; \mu) := \frac{J_0(\mu r)}{J_0(\mu)}, \qquad w = 0, \qquad c = U_r(1; \mu) = \frac{\mu J_0'(\mu)}{J_0(\mu)} = -\frac{\mu J_1(\mu)}{J_0(\mu)}, \tag{2.5}$$

where in the last equality we have used the fact that $J_0' = -J_1$. We recall that, for any $k \geq 0$, the Bessel function J_k is, up to scaling, the unique solution of the ordinary differential equation

$$J_k'' + \frac{1}{\mu}J_k' + \left(1 - \frac{k^2}{\mu^2}\right) J_k = 0 \qquad \text{for } \mu > 0 \tag{2.6}$$

which is finite at $\mu = 0$. We are interested in the solutions (2.5) only for μ lying in the interval I from (2.3), and on this interval both J_0 and J_1 are strictly negative.

2.3. **Simplifying the linear part.** A downside of the formulation (2.4) is that the unknowns u, w are coupled in both (2.4c) and (2.4e). At the nonlinear level this is unavoidable, but at the linear level the equations can be partially decoupled using a standard trick motivated by the chain rule.

Define $v \in C^{2+\alpha}(\overline{\mathbb{D}})$ and $\gamma \in \mathbb{R}$ in terms of $\tilde{u}, w, c$ by

$$\begin{aligned} \tilde{u} &=: U + v + \nabla U \cdot w, \\ c &=: U_r(1;\mu) - \gamma. \end{aligned} \tag{2.7}$$

Inserting (2.7) into (2.4) and grouping the linear terms in v, w, γ, there are several cancellations and we are left with the system

$$\begin{aligned} \partial_1 w_1 - \partial_2 w_2 &= 0 && \text{in } \mathbb{D}, && (2.8\text{a}) \\ \partial_2 w_1 + \partial_1 w_2 &= 0 && \text{in } \mathbb{D}, && (2.8\text{b}) \\ \Delta v + \mu^2 v &= N_1(\nabla v, Dw, \gamma, x; \mu) && \text{in } \mathbb{D}, && (2.8\text{c}) \\ v + U_r\, x \cdot w &= 0 && \text{on } \partial\mathbb{D}, && (2.8\text{d}) \\ U_r v_r - U_{rr} v + U_r \gamma &= N_2(\nabla v, Dw, \gamma, x; \mu) && \text{on } \partial\mathbb{D}, && (2.8\text{e}) \end{aligned}$$

where as a final step we have used (2.8d) to eliminate w from the left hand side of (2.8e). The nonlinear terms are given explicitly by

$$\begin{aligned} N_1(\nabla v, Dw, \gamma, x; \mu) &= -\mu^2 \big(U \det Dw + (v + \nabla U \cdot w) \nabla \cdot w + (v + \nabla U \cdot w) \det Dw \big), \\ N_2(\nabla v, Dw, \gamma, x; \mu) &= \tfrac{1}{2}(v_r^2 - \gamma^2) + U_r(\tfrac{1}{2} v_r + \gamma) \nabla \cdot w + U_{rr} v_r x \cdot w \\ &\quad + \tfrac{1}{2}(\nabla U \cdot w)_r^2 - \tfrac{1}{2} U_r^2 \det Dw + \gamma U_r \det Dw - \tfrac{1}{2}\gamma^2 \nabla \cdot w, \end{aligned}$$

although we emphasize that these formulas play essentially no role in the following analysis.

2.4. **Symmetry and functional setting.** We restrict attention to solutions of (2.8) which are m-fold symmetric in the sense of Theorem 1.2(ii). Identifying $\mathbb{R}^2$ with $\mathbb{C}$, this can be conveniently expressed as

$$v(e^{2\pi i/m} z) = v(z) = v(\bar{z}), \tag{2.9a}$$

$$w(e^{2\pi i/m} z) = e^{2\pi i/m} w(z), \qquad w(\bar{z}) = \overline{w(z)} \tag{2.9b}$$

for all $z \in \overline{\mathbb{D}}$. We also require $Dw(0) = 0$, which by (2.8a), (2.8b), and (2.9b) is equivalent to the single real condition

$$\partial_1 w_1(0) = 0. \tag{2.10}$$

This enforces the normalization $\phi'(0) = 1$ for the conformal mapping $\phi = \mathrm{id} + w$, which in turn fixes the scale of Ω; see Remark 1.5.

We now introduce the Banach spaces

$$\begin{aligned} \mathscr{X} &:= \big\{ (v, w, \gamma) \in C^{2+\alpha}(\overline{\mathbb{D}}) \times C^{2+\alpha}(\overline{\mathbb{D}}, \mathbb{C}) \times \mathbb{R} : (2.8\text{a}),\ (2.8\text{b}),\ (2.9),\ (2.10) \text{ hold} \big\}, \\ \mathscr{Y} &:= \big\{ (f_1, f_2, f_3) \in C^{\alpha}(\overline{\mathbb{D}}) \times C^{2+\alpha}(\partial\mathbb{D}) \times C^{1+\alpha}(\partial\mathbb{D}) : \text{each } f_i \text{ has the symmetry (2.9a)} \big\}. \end{aligned}$$

Defining the interval $I = (j_{1,1}, j_{0,2})$ as in (2.3), we can then interpret (2.8) as an operator equation $\mathscr{F}(v, w, \gamma; \mu) = 0$, where $\mathscr{F} \colon \mathscr{X} \times I \to \mathscr{Y}$ is the real-analytic mapping given by

$$\begin{aligned} \mathscr{F}_1(v, w, \gamma; \mu) &:= \Delta v + \mu^2 v - N_1(\nabla v, Dw, \gamma, x; \mu), \\ \mathscr{F}_2(v, w, \gamma; \mu) &:= v + U_r\, x \cdot w, \\ \mathscr{F}_3(v, w, \gamma; \mu) &:= U_r v_r - U_{rr} v + U_r \gamma - N_2(\nabla v, Dw, \gamma, x; \mu). \end{aligned}$$

Since $\mathscr{F}$ only involves partial derivatives and compositions with real-analytic functions, its real-analyticity is standard; see for instance [dlLO99] and [Val88, proof of Theorem II.5.2]. It is also not difficult to verify that $\mathscr{F}$ respects the symmetries in the definitions of $\mathscr{X}, \mathscr{Y}$, especially if one uses (2.7) to reintroduce $\tilde{u}$ and writes $\det(I + Dw) = |1 + w'|^2$ using complex variables.

The trivial solutions (2.5) are now represented by points $(0, \mu) \in \mathscr{X} \times I$, where the associated linearized operators

$$\mathscr{L}(\mu) := D_{(v,w,\gamma)} \mathscr{F}(0, 0, 0; \mu) \colon \mathscr{X} \longrightarrow \mathscr{Y}$$

are given in components by

$$\mathscr{L}(\mu)\begin{pmatrix} v \\ w \\ \gamma \end{pmatrix} = \begin{pmatrix} \Delta v + \mu^2 v \\ v + U_r\, x \cdot w \\ U_r v_r - U_{rr} v + U_r \gamma \end{pmatrix}. \tag{2.11}$$

The advantage of (2.8) over a more direct approach to (2.4) is that the first and third components of (2.11) have constant coefficients and do not involve w.

3. Proof of the theorem

In this section we prove Theorem 1.2, assuming Lemma 1.1. The main tool is the following real-analytic version of the celebrated Crandall–Rabinowitz theorem [CR71] on bifurcation from a simple eigenvalue.

Theorem 3.1 (Theorem 8.3.1 in [BT03])**.** *Let X, Y be Banach spaces, let $I \subset \mathbb{R}$ be an open interval, and let $F\colon X \times I \to Y$ a real-analytic mapping with $F(0,\mu) = 0$ for all $\mu \in I$. Denoting the associated linearized operators by $L(\mu) := D_x F(0,\mu)$, suppose that, for some $\mu^* \in I$,*

(a) *$L(\mu^*)$ is Fredholm with index 0;*
(b) *the kernel of $L(\mu^*)$ is one dimensional, spanned by $\xi^* \in X$; and*
(c) (Transversality) *$\partial_\mu L(\mu^*)\xi^* \notin \operatorname{ran} L(\mu^*)$.*

Then there exists $\varepsilon_0 > 0$ and a pair of analytic functions $(\tilde{x}, \tilde{\mu})\colon (-\varepsilon_0, \varepsilon_0) \to X \times I$ such that

(i) *$F(\tilde{x}(\varepsilon), \tilde{\mu}(\varepsilon)) = 0$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$;*
(ii) *$\tilde{x}(0) = 0$, $\tilde{\mu}(0) = \mu^*$, and $\tilde{x}_\varepsilon(0) = \xi^*$; and*
(iii) *there exists an open neighborhood U of $(0,\mu^*)$ in $X \times \mathbb{R}$ such that*

$$\big\{(x,\mu) \in U : F(x,\mu) = 0,\ x \neq 0\big\} = \big\{(\tilde{x}(\varepsilon), \tilde{\mu}(\varepsilon)) : 0 < |\varepsilon| < \varepsilon_0\big\}.$$

We will verify the hypotheses of Theorem 3.1 in a series of lemmas. First, we state a basic result about the second component of the operator $\mathscr{L}(\mu)$.

Lemma 3.2. *Suppose that $g \in C^{2+\alpha}(\partial\mathbb{D})$ has the symmetries* (2.9a)*. Then the problem*

$$\begin{aligned} \partial_1 w_1 - \partial_2 w_2 &= 0 \quad \text{in } \mathbb{D}, \\ \partial_2 w_1 + \partial_1 w_2 &= 0 \quad \text{in } \mathbb{D}, \\ x \cdot w &= g \quad \text{on } \partial\mathbb{D} \end{aligned} \tag{3.1}$$

has a solution $w \in C^{2+\alpha}(\overline{\mathbb{D}}, \mathbb{R}^2)$ satisfying (2.9b) *and* (2.10) *if and only if g has mean zero, and in this case the solution is unique.*

Proof. Throughout the proof we identify $\mathbb{R}^2$ with $\mathbb{C}$ whenever convenient. Given g as in the statement, let $G \in C^{2+\alpha}(\overline{\mathbb{D}}, \mathbb{C})$ be the unique solution to the Schwarz boundary value problem

$$\begin{aligned} \partial_1 G_1 - \partial_2 G_2 &= 0 \quad \text{in } \mathbb{D}, \\ \partial_2 G_1 + \partial_1 G_2 &= 0 \quad \text{in } \mathbb{D}, \\ G_1 = \operatorname{Re} G &= g \quad \text{on } \partial\mathbb{D} \end{aligned} \tag{3.2}$$

with $\operatorname{Im} G(0) = 0$. By uniqueness and the symmetry of g, we deduce that G has the symmetry (2.9a). If g has zero mean, then the mean value property for the harmonic function $\operatorname{Re} G$ implies that $\operatorname{Re} G(0) = 0$ and hence $G(0) = 0$. It is now straightforward to check that $w(z) := zG(z)$ solves (3.1) and satisfies (2.9b) and (2.10). Conversely, if w satisfies (3.1) as well as (2.9b) and (2.10), then $G(z) := w(z)/z$ is a well-defined holomorphic function on $\mathbb{D}$ which solves (3.2) with $\operatorname{Im} G(0) = 0$. Uniqueness for (3.2) now gives uniqueness for w, while the mean value property for $\operatorname{Re} G$ and $\operatorname{Re} G(0) = \partial_1 w_1(0) = 0$ together imply that g has zero mean. ☐

Lemma 3.3. *For any $\mu \in I$, the operator $\mathscr{L}(\mu)$ is Fredholm with index 0.*

Proof. As $\mathscr{L}(\mu)$ is a compact perturbation of the operator

$$\tilde{\mathscr{L}}(\mu)\colon \begin{pmatrix} v \\ w \\ \gamma \end{pmatrix} \mapsto \begin{pmatrix} \Delta v - v \\ v + U_r\, x \cdot w \\ U_r v_r \end{pmatrix}, \tag{3.3}$$

it suffices to show that $\tilde{\mathscr{L}}(\mu)$ is Fredholm with index 0. To this end, let $f = (f_1, f_2, f_3) \in \mathscr{Y}$. The first and third components of the equation

$$\tilde{\mathscr{L}}(\mu)(v, w, \gamma) = (f_1, f_2, f_3) \tag{3.4}$$

form an inhomogeneous Neumann problem

$$\begin{aligned} \Delta v - v &= f_1 && \text{in } \mathbb{D}, \\ v_r &= U_r^{-1} f_3 && \text{on } \partial\mathbb{D}. \end{aligned}$$

This problem has a unique solution $v = \tilde{v}(f_1, f_3) \in C^{2+\alpha}(\overline{\mathbb{D}})$, and uniqueness together with the symmetries of the data f_1, f_3 in the definition of $\mathscr{Y}$ forces v to satisfy (2.9a).

It remains to consider the second component of (3.4), which rearranges to

$$x \cdot w = U_r^{-1}(f_2 - \tilde{v}(f_1, f_3)) \qquad \text{on } \partial\mathbb{D}.$$

Applying Lemma 3.2, we deduce that (3.4) has a solution if and only if the solvability condition

$$\int_{\partial\mathbb{D}} (f_2 - \tilde{v}(f_1, f_3))\, ds = 0$$

holds, and that in this case the solution is unique. Since the parameter $\gamma \in \mathbb{R}$ does not appear on the right hand side of (3.3), we conclude that $\tilde{\mathscr{L}}(\mu)$ has index zero and the proof is complete. □

Lemma 3.4. *Let μ_m be as in Lemma 1.1. Then the kernel $\mathscr{L}(\mu_m)$ is one-dimensional, spanned by $(V_m(\,\cdot\,;\mu_m), W_m(\,\cdot\,;\mu_m), 0)$ where V_k, W_k are given by the formulas*

$$\begin{aligned} V_k(re^{i\theta};\mu) &:= J_k(\mu r)\cos k\theta, \\ W_k(re^{i\theta};\mu) &:= \frac{J_k(\mu)J_0(\mu)}{\mu J_1(\mu)} r^{k+1} \begin{pmatrix} \cos((k+1)\theta) \\ \sin((k+1)\theta) \end{pmatrix}. \end{aligned} \tag{3.5}$$

Proof. Let $\mu \in I$ and suppose that $(v, w, \gamma) \in \mathscr{X}$ lies in the kernel of $\mathscr{L}(\mu)$, i.e.

$$\Delta v + \mu^2 v = 0 \qquad \text{in } \mathbb{D}, \tag{3.6a}$$

$$v + U_r\, x \cdot w = 0 \qquad \text{on } \partial\mathbb{D}, \tag{3.6b}$$

$$U_r v_r - U_{rr} v + U_r \gamma = 0 \qquad \text{on } \partial\mathbb{D}. \tag{3.6c}$$

If $v \equiv 0$, then Lemma 3.2 and (3.6c) immediately imply that $w \equiv 0$ and $\gamma = 0$ as well.

Suppose then that $v \not\equiv 0$. From (3.6a) and familiar Fourier series arguments, it is enough to consider the case where $v = V_k$ for some $k \geq 0$ which is an integer multiple of m. If $k = 0$, then $v = J_0(\mu_m) < 0$ on $\partial\mathbb{D}$, and so (3.6b) violates the solvability condition in Lemma 3.2. Thus we can restrict our attention to $k \geq 1$. By Lemma 3.2, the boundary condition (3.6b) can then be uniquely solved for w, and an easy calculation confirms that this solution is $w = W_k$. Averaging (3.6c) yields $\gamma = 0$.

Substituting $v = V_k$, $w = W_k$, and $\gamma = 0$ in (3.6c) and recalling (2.5), we are finally left with

$$U_r V_r - U_{rr} V = -\frac{\mu^2}{J_0(\mu)} \mathcal{W}_{1,k}(\mu) \cos k\theta = 0 \quad \text{ on } \partial\mathbb{D},$$

where $\mathcal{W}_{1,k} = J_1 J_k' - J_k J_1'$ is the Wronskian from Lemma 1.1. Setting $\mu = \mu_m$, we have from the same lemma that $\mathcal{W}_{1,m}(\mu_m) = 0$, and moreover that $\mathcal{W}_{1,k}(\mu_m) < 0$ for all integers $k > m$. The proof is complete. □

Lemma 3.5. *There are no solutions $(v, w, \gamma) \in \mathscr{X}$ to the equation*

$$\mathscr{L}(\mu_m)(v, w, \gamma) + \partial_\mu \mathscr{L}(\mu_m)(V_m(\,\cdot\,;\mu_m), W_m(\,\cdot\,;\mu_m), 0) = 0. \tag{3.7}$$

Proof. To simplify the notation, we suppress the first arguments of the functions V_m, W_m defined in (3.5). We have seen in the proof of Lemma 3.4 that, for any $\mu \in I$, $(V_m(\mu), W_m(\mu), 0)$ satisfies

$$\mathscr{L}(\mu) \begin{pmatrix} V_m(\mu) \\ W_m(\mu) \\ 0 \end{pmatrix} = -\frac{\mu^2}{J_0(\mu)} \mathcal{W}_{1,m}(\mu) \begin{pmatrix} 0 \\ 0 \\ \cos m\theta \end{pmatrix}.$$

Differentiating this identity with respect to μ and using the fact that $\mathcal{W}_{1,m}(\mu_m) = 0$, we find

$$\mathscr{L}(\mu)\begin{pmatrix}\partial_\mu V_m(\mu_m)\\ \partial_\mu W_m(\mu_m)\\ 0\end{pmatrix} + \partial_\mu \mathscr{L}(\mu_m)\begin{pmatrix}V_m(\mu_m)\\ W_m(\mu_m)\\ 0\end{pmatrix} = \frac{\mu_m^2}{J_0(\mu_m)}\mathcal{W}_{1,m}'(\mu_m)\begin{pmatrix}0\\ 0\\ \cos m\theta\end{pmatrix}. \tag{3.8}$$

Suppose now that $(v, w, \gamma) \in \mathscr{X}$ solves (3.7). Subtracting (3.8), we find

$$\mathscr{L}(\mu)\begin{pmatrix}v - \partial_\mu V_m(\mu_m)\\ w - \partial_\mu W_m(\mu_m)\\ \gamma\end{pmatrix} = -\frac{\mu_m^2}{J_0(\mu_m)}\mathcal{W}_{1,m}'(\mu_m)\begin{pmatrix}0\\ 0\\ \cos m\theta\end{pmatrix}. \tag{3.9}$$

By Lemma 1.1, $\mathcal{W}_{1,m}'(\mu_m) \neq 0$. Arguing exactly as in the proof of Lemma 3.4, we conclude that (3.9) has no solutions, and hence that the same is true for (3.7). □

Proof of Theorem 1.2. We apply Theorem 3.1 with $F = \mathscr{F}$, $X = \mathscr{X}$, $Y = \mathscr{Y}$, and $\mu^* = \mu_m$. The hypotheses (a), (b), (c) have been verified in Lemmas 3.3, 3.4, and 3.5, respectively. Thus there exists a family of solutions

$$\big(v(\varepsilon), w(\varepsilon), \gamma(\varepsilon), \mu(\varepsilon)\big) \in \mathscr{X} \times I$$

to (2.8), parameterized by $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and with the asymptotic expansions

$$\begin{aligned} v(\varepsilon) &= \varepsilon V_m(\mu_m) + O(\varepsilon^2),\\ w(\varepsilon) &= \varepsilon W_m(\mu_m) + O(\varepsilon^2),\\ \gamma(\varepsilon) &= O(\varepsilon^2),\\ \mu(\varepsilon) &= \mu_m + O(\varepsilon) \end{aligned}$$

as $\varepsilon \to 0$. Here for notational convenience we are suppressing the first arguments of v, w, V_m, W_m. The improved expansion $\mu(\varepsilon) = \mu_m + O(\varepsilon^2)$ for the parameter follows as usual from the symmetry of $\mathscr{F}$ under rotations by an angle π/m; see for instance [Kie12, discussion leading to (I.14.41)].

These expansions in particular imply $\|w(\varepsilon)\|_{C^{2+\alpha}} = O(\varepsilon)$ and so, perhaps after shrinking $\varepsilon_0 > 0$, the map $\phi(\varepsilon) = \mathrm{id} + w(\varepsilon)$ is a $C^{2+\alpha}$ diffeomorphism, and its image $\Omega(\varepsilon) := \phi(\mathbb{D}; \varepsilon)$ is a $C^{2+\alpha}$ domain. Thus, defining $u(\varepsilon) \in C^{2+\alpha}(\overline{\mathbb{D}})$ using (2.7), i.e.

$$u(\varepsilon) := \tilde{u}(\varepsilon) \circ \phi(\varepsilon)^{-1} \quad \text{where} \quad \tilde{u}(\varepsilon) := U(\mu(\varepsilon)) + v(\varepsilon) + \nabla U(\mu(\varepsilon)) \cdot w(\varepsilon),$$

and defining $c(\varepsilon) := U_r(1; \mu(\varepsilon)) - \gamma(\varepsilon)$ and $\lambda(\varepsilon) := (\mu(\varepsilon))^2$, we obtain the desired solutions of the original problem (1.1) with $b = 1$. Here we must check, though, that we have chosen the correct sign for c, since in passing from (1.1) to (2.8) we replaced (1.1c) with (2.1). If necessary, further shrink $\varepsilon_0 > 0$ so that $c(\varepsilon) < 0$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Inspecting the trivial solution at $\varepsilon = 0$, we see that the sign of $c(0)$ was indeed chosen correctly, and hence by a continuity argument that it is correct for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Finally, inserting the asymptotic expansions above into (2.7) using the explicit formulas in (2.5) and (3.5) yields the claimed expansion (1.4), where here we have rescaled ε by a factor of $J_m(\mu_m)J_0(\mu_m)/(\mu J_1(\mu_m))$ to simplify the form of (1.4a). □

4. Proof of Lemma 1.1

This final section is devoted to the proof of Lemma 1.1. First we introduce, for general integers $k, \ell \geq 0$, the Wronskians

$$\mathcal{W}_{k,\ell} := J_k J_\ell' - J_\ell J_k'. \tag{4.1}$$

As noted in [Pá13, Section 3.2], an immediate consequence of the differential equations (2.6) satisfied by J_k, J_ℓ is that these Wronskians satisfy

$$\frac{d}{d\mu}\big(\mu \mathcal{W}_{k,\ell}(\mu)\big) = \frac{\ell^2 - k^2}{\mu} J_k(\mu) J_\ell(\mu). \tag{4.2}$$

This suggests that the roots of $\mathcal{W}_{k,\ell}$ are closely related to the interlacing of the roots of J_k, J_ℓ; see [Pá13, Lemma 5]. Since Lemma 1.1 concerns the roots of $\mathcal{W}_{1,m}$, we are therefore particularly interested in understanding how the roots of J_1 and J_m interlace.

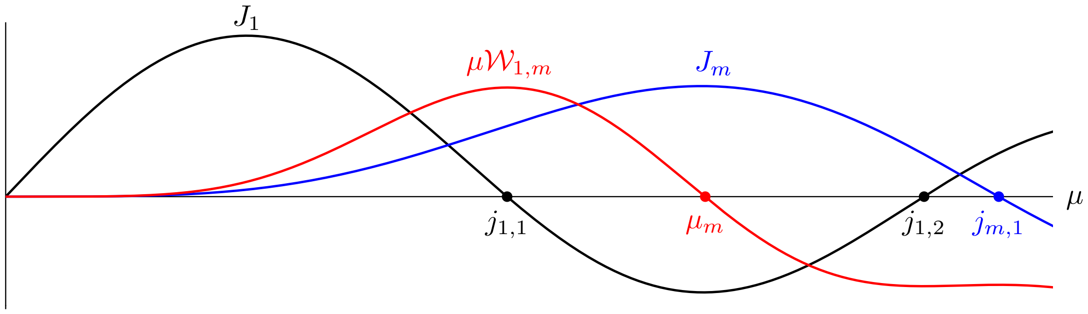


FIGURE 2. Plots of $J_1(\mu), J_m(\mu), \mu\mathcal{W}_{1,m}(\mu)$ for $m = 4$. Their roots satisfy the inequalities $j_{1,1} < \mu_m < j_{1,2} < j_{m,1}$, and $\mu\mathcal{W}_{1,m}$ is strictly decreasing on the interval $(j_{1,1}, j_{1,2})$.

For integers $m \geq 0$, let $j_{m,1} < j_{m,2} < \cdots$ denote the positive roots of J_m, all of which are simple. Importantly, $j_{m,n}$ is a strictly increasing function of m for each n. From the numerical values $j_{3,1} \approx 6.3802$, $j_{1,2} \approx 7.0156$, and $j_{4,1} \approx 7.5883$, we see that $j_{3,1} < j_{1,2} < j_{4,1}$, and so this monotonicity implies

$$j_{1,2} < j_{m,1} \quad \text{for all } m \geq 4. \tag{4.3}$$

The inequality (4.3) is the reason for the restriction $m \geq 4$ in Lemma 1.1. Indeed, the result is false for $m = 0, 1, 2, 3$.

Proof of Lemma 1.1. Fix an integer $m \geq 4$. The functions J_1, J_m vanish at the origin and are positive for small positive arguments. This together with (4.3) gives the inequalities

$$\begin{gathered} J_1 > 0 \text{ on } (0, j_{1,1}), \quad J_1 < 0 \text{ on } (j_{1,1}, j_{1,2}), \quad J_m > 0 \text{ on } (0, j_{1,2}), \\ J_1'(j_{1,1}) < 0 < J_1'(j_{1,2}) \end{gathered} \tag{4.4}$$

illustrated in Figure 2. From the definition of $\mathcal{W}_{1,m}$ we immediately conclude that

$$\mathcal{W}_{1,m}(0) = 0, \quad \mathcal{W}_{1,m}(j_{1,1}) > 0 > \mathcal{W}_{1,m}(j_{1,2}), \tag{4.5}$$

while (4.2) and (4.4) yield

$$(\mu\mathcal{W}_{1,m})' > 0 \text{ on } (0, j_{1,1}), \quad (\mu\mathcal{W}_{1,m})' < 0 \text{ on } (j_{1,1}, j_{1,2}). \tag{4.6}$$

From (4.5) and (4.6) it is clear that $\mathcal{W}_{1,m}$ has a root $\mu_m \in (j_{1,1}, j_{1,2})$, that this root is simple, and that there are no other roots in the interval $(0, j_{1,2})$.

Next we show that $\mu_k < \mu_m$ for $k > m \geq 4$. By (4.2) and (4.4), the function $\mu\mathcal{W}_{k,m}$ is strictly decreasing on $(0, j_{1,2})$, so that in particular $\mathcal{W}_{k,m}(\mu_m) < 0$. Using the algebraic identity

$$J_1\mathcal{W}_{k,m} - J_k\mathcal{W}_{1,m} + J_m\mathcal{W}_{1,k} = 0$$

and the fact that $\mathcal{W}_{1,m}(\mu_m) = 0$, we therefore find

$$\mathcal{W}_{1,k} = -\frac{J_1}{J_m}\mathcal{W}_{k,m} < 0 \quad \text{at } \mu_m.$$

Since (4.6) holds for $m = k$ and $\mathcal{W}_{1,k}(\mu_k) = 0$, we deduce that $\mu_k < \mu_m$ as desired.

It remains to show that $\mu_m < j_{0,2}$. Since $j_{0,1} < j_{1,1} < j_{0,2} < j_{1,2}$, this will in particular imply that $J_0(\mu_m) < 0$. As μ_m is a decreasing function of $m \geq 4$, it suffices to show $\mu_4 < j_{0,2}$, and arguing as above this is equivalent to the inequality $\mathcal{W}_{1,4}(j_{0,2}) < 0$. Using a computer algebra system we numerically calculate $\mathcal{W}_{1,4}(j_{0,2}) \approx -0.012148 < 0$, and the proof is complete. □

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Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom
Email address: mw2319@bath.ac.uk