Some basic results and definitions from Chapter 1

Theorem 1.4 (Cauchy–Schwarz inequality). *Let $(X, \langle\cdot\,,\cdot\rangle)$ be an inner product space. Then $|\langle x, y\rangle| \le \|x\|\|y\|$ for all $x, y \in X$.*

Definition 1.9. Suppose that (X, d) is a metric space and $Y \subset X$ is a subset, and let $d' = d|_{Y\times Y}$. Then (Y, d') is a metric space and is called a *metric subspace* of (X, d).

Definition 1.18. Let (X, d) be a metric space. A set $U \subseteq X$ is called *open* if for every $x \in U$ there exists $r > 0$ such that $B_r(x) \subseteq U$. A set $F \subseteq X$ is called *closed* if $X \setminus F$ is open.

Definition 1.20. Let (X, d) be a metric space and $S \subseteq X$. The *interior* S° of S is the set of all $s \in S$ with the property that there exists $r > 0$ such that $B_r(s) \subseteq S$. The *closure* $\overline{S}$ of S is the set of all $x \in X$ such that $B_r(x) \cap S \neq \varnothing$ for any $r > 0$.

Definition 1.29. Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. A point $x_0 \in X$ is said to be a *limit* of the sequence if $d(x_n, x_0) \to 0$ in $\mathbb{R}$ as $n \to \infty$. If so, we say that the sequence *converges* to x_0 and we write $x_0 = \lim_{n\to\infty} x_n$ or $x_n \to x_0$.

Theorem 1.35. *Let (X, d) be a metric space. A set $S \subseteq X$ is closed if and only if, for every sequence in S that converges in X, the limit belongs to S.*

Definition 1.36. A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is called a *Cauchy sequence* if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \ge N$.

Definition 1.38. A metric space (X, d) is called *complete* if every Cauchy sequence in X is convergent. A complete normed space is called a *Banach space* and a complete inner product space is called a *Hilbert space.*

Theorem 1.41. *Let (X, d) be a complete metric space and let (Y, d') be a metric subspace. Then (Y, d') is complete if and only if Y is closed as a subset of X.*

Definition 1.43. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is called *uniformly continuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $x_1, x_2 \in X$, $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \varepsilon$.

Theorem 1.46 (Sequential continuity). *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $x_0 \in X$ and let $f\colon X \to Y$ be a map. The following are equivalent.* (i) *The map f is continuous at x_0.* (ii) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X with $x_n \to x_0$, then $f(x_n) \to f(x_0)$.*

Definition 1.48. Let (X, d) be a metric space. Then $C_{\mathrm{b}}(X)$ denotes the space of continuous, bounded functions $f\colon X \to \mathbb{R}$, equipped with the supremum norm.

Theorem 1.50. *For any metric space (X, d), the space $C_{\mathrm{b}}(X)$ is a Banach space.*

Corollary. *For any $a, b \in \mathbb{R}$ with $a < b$, the space $C^0([a, b]) = C_{\mathrm{b}}([a, b])$ is a Banach space.*

Definition 1.62. Let (X, d_X) and (Y, d_Y) be metric spaces. If $f\colon X \to Y$ is a map such that $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$, then f is called an *isometry.* The two metric spaces are called *isometric* if there exists a bijective isometry between them.