Solitary water waves of large amplitude generated by surface pressure

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Abstract

We consider exact nonlinear solitary water waves on a shear flow with an arbitrary distribution of vorticity. Ignoring surface tension, we impose a non-constant pressure on the free surface. Starting from a uniform shear flow with a flat free surface and a supercritical wave speed, we vary the surface pressure and use a continuation argument to construct a global connected set of symmetric solitary waves. This set includes waves of depression whose profiles increase monotonically from a central trough where the surface pressure is at its lowest, as well as waves of elevation whose profiles decrease monotonically from a central crest where the surface pressure is at its highest. There may also be two waves in this connected set with identical surface pressure, only one of which is a wave of depression.

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1 Introduction

1.1 Informal discussion of results

We consider steady traveling waves in a two-dimensional, incompressible, inviscid fluid under the influence of gravity. The fluid is bounded below by a horizontal bed and above by a free surface. In addition to the usual kinematic boundary conditions, we impose a non-constant pressure on the free surface which tends to a constant "atmospheric" value at infinity. Such a surface pressure can be used, for instance, to model the influence of a ship. We consider the effect of surface tension to be negligible.

Localized pressure disturbances on the free surfaces of uniform flows have been well-studied in the linearized irrotational literature [Sto92, KMV02]. The effect of the disturbance depends on the *Froude number* F, which is a dimensionless measure of the wave speed c. For *supercritical* Froude numbers $F > 1$, the free surface is asymptotically flat at infinity. For *subcritical* Froude numbers $F < 1$, on the other hand, there may be periodic behavior far upstream or downstream, though the waves which are expected to occur in reality are only periodic downstream [Bea80].

In this paper, we construct nonlinear solitary waves with large amplitude under the assumption that the Froude number is supercritical. We allow for an arbitrary distribution of vorticity; at infinity these waves converge to a uniform shear flow whose horizontal velocity U is not necessarily constant but may depend on the vertical variable y. Our definition of the Froude number for waves with vorticity is given in (1.2). We also assume that both the free surface and surface pressure are symmetric, and that the surface pressure has at most one local extremum. To find large-amplitude waves, we start from a uniform shear flow and apply a degree-theoretic continuation argument in which the surface pressure is varied while the asymptotic shear flow $U(y)$ and Froude number F are held fixed.

For small-amplitude waves, the relationship between the surface pressure and the height of the free surface depends on the Froude number. William Thomson (Lord Kelvin) gives an elegant account of this relationship in an 1887 lecture [Tho87]:

> "Now to find mathematically the velocity of progression of a free wave, proceed thus. Take your gutta-percha form [model ship hull] and hold it stationary on the surface of the water; the water-pressure is less at the crest and greater at the hollow; by the law of hydrostatics, the deeper down you go, the greater is the pressure. Move your form along very rapidly, and a certain result, a centrifugal force, due to the inertia of the flowing water, will now cause the pressure to be greatest at the crest and least at the lowest point of the hollow. Move it along at exactly the proper speed, and you will cause the pressure to be equal all over the surface of the gutta-percha form. We only had it in imagination. Having imagined it and got what we wanted out of it, discard it when moving at exactly this proper speed, and then you have a free wave."

For slower (subcritical) waves, we expect the surface pressure to be lower at crests and higher at troughs. For the "very rapid" (supercritical) waves which we consider in this paper, however, we expect the reverse: the surface pressure is higher at crests and lower at troughs (see Corollaries 2.5 and 2.6 for more precise statements).

The waves that we construct fall into three categories. The first two, which always exist, are solitary waves of elevation (depression) whose free surfaces decrease (increase) monotonically from a central crest (trough), see Theorem 1.2. As for small-amplitude supercritical waves, the surface pressures for these waves are higher at crests and lower at troughs. Under some assumptions on the waves of elevation, including the existence of a nontrivial "free wave" with constant surface pressure, there also exist waves of a third type, see Theorem 1.4. Some of these waves have the same surface pressures as the waves of depression mentioned above, but unlike those waves of depression have one or more tall crests. Analogues of these three types of solutions have been observed numerically for a related problem [AVB94].

1.2 Statement of the main results

We denote the horizontal bed by $y = 0$ and the free surface by $y = \eta(x, t)$. Fixing the constant wave speed $c > 0$, we assume that the motion is steady in that the free surface η, velocity field (u, v), and pressure P depend only on $x - ct$ and y. Relabeling $x - ct$ as x, (u, v) and P satisfy the stationary Euler equations

$$(u-c)u_x + vu_y = -P_x, \qquad (u-c)v_x + vu_y = -P_y - g, \qquad u_x + v_y = 0, \tag{1.1a}$$

in the fluid domain $D_\eta = \{(x, y) : -d < y < \eta(x)\}$, together with the usual kinematic boundary conditions

$$v = 0 \ \text{ on } y = 0, \qquad v = (u-c)\eta_x \ \text{ on } y = \eta(x). \tag{1.1b}$$

Ignoring the effect of surface tension, we also prescribe the pressure on the free surface,

$$P = P_{\text{atm}} + R(x) \ \text{ on } y = \eta(x). \tag{1.1c}$$

Here $R(x)$ vanishes as $x \to \pm\infty$, but is not necessarily small, and represents the deviation of the surface pressure from the (constant) atmospheric pressure P_{atm} at $x = \pm\infty$. In the classical water wave problem, $R \equiv 0$. We also impose asymptotic conditions

$$\eta \to 0, \quad v \to 0, \quad u \to U(y), \qquad \text{as } x \to \pm\infty, \tag{1.1d}$$

uniformly in y, where $U(y)$ is the horizontal velocity of the shear flow at $x = \pm\infty$. The requirement that η vanish at infinity ensures that d is the asymptotic depth of the fluid.

Assuming that $U(y) < c$, we define the Froude number F by

$$\frac{1}{F^2} = g \int_{-d}^{0} \frac{dy}{(c - U(y))^2}. \tag{1.2}$$

In this paper we will only consider supercritical Froude numbers $F > 1$. We assume further that $u < c$ throughout the fluid, which we call our *no-stagnation assumption* because it rules out the existence of points (x, y), called *stagnation points*, where $(u, v) = (c, 0)$. We will deal primarily with *symmetric* waves, that is waves where u, η, and R are even in x while v is odd in x. We call a wave *trivial* if $\eta \equiv 0$, $v \equiv 0$, and $u \equiv U(y)$. A *wave of elevation* is one with $\eta(x) > 0$ for all $x \in \mathbb{R}$, while a *wave of depression* has $\eta(x) < 0$ for all x. We call a symmetric wave *monotone* if η and η' have opposite signs for $x \neq 0$. We call a wave a *free wave* if it has $R \equiv 0$.

For the purposes of continuation, we introduce a pressure parameter $\beta \in \mathbb{R}$ and consider a 1-parameter family $R = R(x; \beta)$ of surface pressures. A simple example is $R(x; \beta) = \beta R^0(x)$ where $R^0 \in L^1(\mathbb{R})$ is even, nonnegative, and monotone decreasing for $x \geq 0$. More generally, $R(x; \beta)$ is a family of even functions with $R(x; 0) \equiv 0$ and

$$x\beta R_x(x; \beta) \leq 0, \qquad \lim_{\beta \to +\infty} R(0; \beta) = +\infty, \qquad \int_{\mathbb{R}} \sup_{|\beta| < M} |R(x; \beta)| \, dx < \infty, \tag{1.3}$$

for any $M > 0$. Note that we make no assumptions about the behavior of R as $\beta \to -\infty$. One physically meaningful choice would be for the surface pressure $R(0;\beta)$ at the origin to tend to $-\infty$; another would be for the force $\int_{\mathbb{R}} R(x;\beta)\,dx$ exerted by R on a flat free surface to tend to $-\infty$.

To state our main results, we now fix the constants $g, c, d > 0$ and a Hölder parameter $\alpha \in (0,1)$. We fix a shear flow $U \in C^{3+\alpha}[-d,0]$ with $\max U < c$ and $F > 1$, and also a 1-parameter family $R(x;\beta)$ as above with $\beta \mapsto R(\,\cdot\,;\beta)$ in $C^2(\mathbb{R}, C^{2+\alpha}_{\mathrm{b}}(\mathbb{R}))$. We then seek symmetric solutions

$$(\beta, u, v, \eta) \in \mathbb{R} \times C^{2+\alpha}_{\mathrm{b}}(\overline{D_\eta}) \times C^{2+\alpha}_{\mathrm{b}}(\overline{D_\eta}) \times C^{3+\alpha}_{\mathrm{b}}(\mathbb{R})$$

of (1.1). Our first result is the following:

Theorem 1.1. *The trivial solution $(\beta, u, v, \eta) = (0, U, 0, 0)$ of (1.1) lies on a unique C^2 curve of solutions $\mathscr{C}_1$:*

$$(u, v, \eta) = (u, v, \eta)(\beta), \qquad \beta_- < \beta < \beta_+, \qquad -\infty \le \beta_- < 0 < \beta_+ < \infty, \tag{1.4}$$

with the following properties. Solutions in $\mathscr{C}_1$ with $\beta > 0$ are monotone waves of elevation, those with $\beta < 0$ are monotone waves of depression, and all solutions in $\mathscr{C}_1$ satisfy

$$g \sup_x \int_{-d}^{\eta(x)} \frac{dy}{(c - u(x,y))^2} < 1. \tag{1.5}$$

Moreover, as $\beta \to \beta_+$ along $\mathscr{C}_1$ either (1.5) tends to an equality or $\sup_{D_\eta} u \to c$. As $\beta \to \beta_-$ we have the same two alternatives or else $\beta_- = -\infty$.

We call the possibility $\sup_{D_\eta} u \to c$ in Theorem 1.1 *stagnation*; it means there are waves in $\mathscr{C}_1$ which come arbitrarily close to violating our assumption $u < c$. We make no claim that v is simultaneously near 0. We do not give a physical interpretation of (1.5), but do note its similarity to the definition (1.2) of the Froude number F. Indeed, the trivial solution $(0, U, 0, 0)$ satisfies (1.5) if and only if $F > 1$. If $\beta_- = -\infty$, then we have found solutions for all negative values of β. The significance of these solutions depends on the behavior of R as $\beta \to -\infty$, which we have not specified. In Appendix A.1, we consider an explicit family of solutions of (1.1) with constant vorticity. This family includes sequences of waves of depression where the surface pressure $R(0)$ at the central trough tends to $-\infty$ while the height $\eta(0) + d$ of the trough above the bed tends to 0. It also includes waves of elevation arbitrarily close to stagnation.

In our next result, we completely remove the restriction (1.5) and extend $\mathscr{C}_1$ to a possibly larger connected set $\mathscr{C}_2^- \cup \mathscr{C}_2^+$ of solutions. The monotonicity properties of $\mathscr{C}_1$ are preserved, but $\mathscr{C}_2^\pm$ are no longer necessarily curves. Along $\mathscr{C}_2^+$ there is also a new alternative: there exists a nontrivial free wave in $\mathscr{C}_2^+$, that is a wave with constant pressure $P = P_{\mathrm{atm}}$ on its free surface.

Theorem 1.2. *The set $\mathscr{C}_1^- = \mathscr{C}_1 \cap \{\beta < 0\}$ is contained in a connected set $\mathscr{C}_2^-$ of monotone waves of depression with $\beta < 0$ satisfying one of the two alternatives*

(i$^-$) *(Stagnation)* $\sup\limits_{\mathscr{C}_2^-} \sup\limits_{D_\eta} u = c$*; or*

(ii$^-$) *(β large and negative)* $\inf\limits_{\mathscr{C}_2^-} \beta = -\infty$.

The set $\mathscr{C}_1^+ = \mathscr{C}_1 \cap \{\beta > 0\}$ is contained in a connected set $\mathscr{C}_2^+$ of monotone waves of elevation with $\beta \ge 0$ satisfying one of the two alternatives

(i$^+$) *(Stagnation)* $\sup\limits_{\mathscr{C}_2^+} \sup\limits_{D_\eta} u = c$*; or*

(ii$^+$) *(Free wave) There exists a solution other than $(0, U, 0, 0)$ in $\mathscr{C}_2^+$ with $\beta = 0$.*

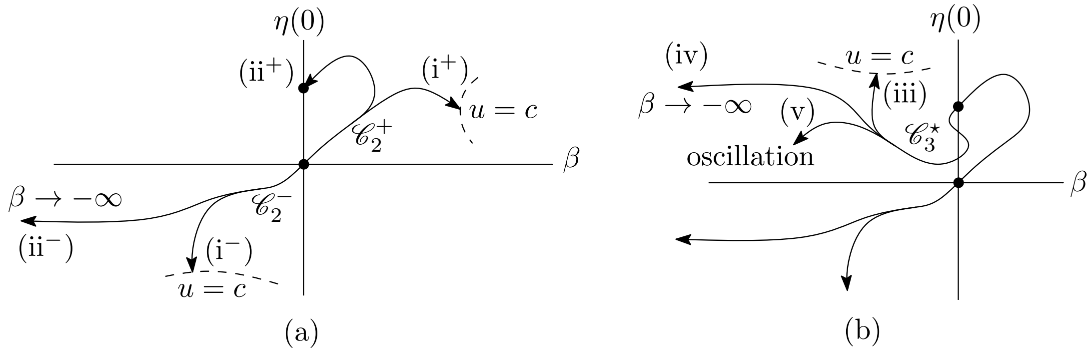

Figure 1: (a) The alternatives in Theorem 1.2. (b) The alternatives in Theorem 1.4.

The various alternatives in Theorem 1.2 are illustrated in Figure 1a. Comparing Theorem 1.2 with Theorem 1.1, we see that the restriction (1.5) has been completely eliminated for $\mathscr{C}_2^-$, while for $\mathscr{C}_2^+$ it is replaced by the new alternative (ii$^+$). At least in some special cases, we expect that alternative (i$^-$) can be eliminated, leaving the single alternative (ii$^-$) for $\mathscr{C}_2^-$. In Section 1.4 we give a partial result in this direction: If the shear flow U satisfies certain conditions, then alternative (i$^-$) implies that the free surface becomes vertical along $\mathscr{C}_2^-$.

In our last result we consider what happens to the free wave in alternative (ii$^+$) as β is decreased below 0. It seems unlikely that it will remain monotone. One possibility is that the central crest splits into two humps, and that along some sequence of solutions these humps translate to $x = \pm\infty$. This behavior was observed numerically in [AVB94] for a related problem. To describe this and similar situations we make the following definition:

Definition 1.3 (Oscillatory sequence)**.** We call a sequence of symmetric solutions $(\beta_n, u_n, v_n, \eta_n)$ of (1.1) *non-oscillatory* if there exists $M > 0$ such that each v_n does not change sign for $x \geq M$. Otherwise we call $(\beta_n, u_n, v_n, \eta_n)$ *oscillatory.*

Note that, since $v = (u - c)\eta_x$ on the free surface and $u < c$ throughout the fluid, the sign of η_x is determined by the sign of v on the free surface.

Theorem 1.4. *Consider the same situation as in Theorem 1.2. Suppose that alternative* (i$^+$) *does not hold, so that $\mathscr{C}_2^+$ meets $\{\beta = 0\}$. Then $\mathscr{C}_2^+$ is contained in a bigger connected set $\mathscr{C}_3^\star$ of (possibly non-monotone) solutions such that*

(a) *Some waves in $\mathscr{C}_3^\star$ have $\beta < 0$. The maximum heights $d + \max \eta$ above the bottom of all waves in $\mathscr{C}_3^\star \cap \{\beta < 0\}$ are bounded below by d^*. The depth d^* is defined in Section 2.1, and is greater than d. In particular, $\mathscr{C}_3^\star$ does not meet $\mathscr{C}_2^-$.*

(b) *One of the following three alternatives holds:*

(iii) *(Stagnation)* $\sup\limits_{\mathscr{C}_3^\star \setminus \mathscr{C}_2^+} \sup\limits_{D_\eta} u = c$*; or*

(iv) *(β large and negative)* $\inf\limits_{\mathscr{C}_3^\star \setminus \mathscr{C}_2^+} \beta = -\infty$*; or*

(v) *(Oscillation) $\mathscr{C}_3^\star \setminus \mathscr{C}_2^+$ contains an oscillatory sequence $(\beta_n, u_n, v_n, \eta_n)$.*

Thus in Theorem 1.2, we can replace the single alternative (ii$^+$) for $\mathscr{C}_2^+$ by the three alternatives (iii), (iv), and (v) for $\mathscr{C}_3^\star$, which are illustrated in Figure 1b. Notice in part (a) of the theorem that, unlike waves in $\mathscr{C}_2^-$, waves in $\mathscr{C}_3^\star$ with $\beta < 0$ are *not* waves of depression. We remark that results similar to Theorems 1.2 and 1.4 can be obtained by continuing from a nontrivial free wave $(0, u, v, \eta)$

which is non-degenerate in the sense that the corresponding linearized operator is invertible. Such a wave is guaranteed to exist when the Froude number F is sufficiently close to 1 [Whe, Hur08a, GW08].

We now compare the above results with [AVB94]. In that paper, Asavanant and Vanden-Broeck numerically compute irrotational waves which have a parabolic obstacle with equation $y = \frac{1}{2}\epsilon + y_0$ lying on their free surface. The points $\pm L/2$ where the free surface separates from the obstacle are additional unknowns in the problem. These waves will generally have non-constant surface pressure $R(x)$ for $|x| < L/2$, but this is not computed in the paper. In particular, we do not know if R satisfies the monotonicity assumption in (1.3).

Asavanant and Vanden-Broeck identify three families of supercritical waves. Waves of the first type have a concave object resting below the level of the free surface at infinity ($\epsilon > 0$, $y_0 < 0$). From the figures they appear to be monotone waves of depression, and they approach uniform flows as $\epsilon \to 0$. Thus solutions of this first type are analogous to those in $\mathscr{C}_2^-$. Solutions of the second type have a concave object resting above the level of the free surface at infinity ($\epsilon > 0$, $y_0 > 0$). As $\epsilon \to 0$, they do not approach uniform flows. Some of these solutions are perturbations of nontrivial flows with $\epsilon = 0$. Others have free surfaces with two 'bumps' surrounding a central 'valley'. As $\epsilon \to 0$, these bumps translate to $\pm\infty$ without decreasing in amplitude, forming an oscillatory sequence according to Definition 3.9. Thus waves of this second type are analogous to those in $\mathscr{C}_3^\star \setminus \mathscr{C}_2^+$. Finally, there is a third type of solution, with a convex object resting above the level of the free surface at infinity ($\epsilon < 0$, $y_0 > 0$). From the figures they appear to be monotone waves of elevation. Some of these solutions are perturbations of a uniform flow, while others are perturbations of nontrivial flows with $\epsilon = 0$. Thus waves of this third type are analogous to those in $\mathscr{C}_2^+$.

In 1980, Beale [Bea80] constructed exact small-amplitude subcritical waves with a pressure disturbance as perturbations of a uniform flow. One of the main difficulties is the periodic behavior downstream. Small-amplitude waves with a pressure disturbance and F slightly greater than 1 were constructed by Mielke [Mie86, Mie88] using spatial dynamics methods, and later by Sun and Shen [SS93] as part of a rigorous two-parameter asymptotic expansion. These waves are related to the celebrated KdV-type solitary water waves, and are found after rescaling the horizontal variable x. In a series of papers Pagani and Pierotti constructed small-amplitude waves with a semi-submerged object in the supercritical [PP99a, PP99b, PP00, PP01] and later subcritical [PP04] regimes. For a semi-submerged object, the boundary condition changes type where the free surface separates from the object, and these separation points are additional unknowns in the problem. The waves in all of the above references are irrotational.

To the best of our knowledge, the only results concerning large-amplitude waves generated by a non-constant surface pressure or semi-submerged object are numerical. Părău and Vanden-Broeck [PVB02] computed flows due to localized pressure disturbances in water of infinite depth. As in the subcritical case with finite depth, these waves are asymptotically periodic downstream. Părău and Vanden-Broeck also explain how such a method can be used in an inverse way to compute flows past a semi-submerged object. In [VBK89], Vanden-Broeck and Keller constructed solitary waves of elevation with a flat "surfboard" riding on the free surface. Later, Asavanant and Vanden-Broeck considered the more general case of parabolic objects [AVB94] discussed above. Again, the waves in all of these references are irrotational.

Before giving an outline, we compare the current paper to [Whe], in which we constructed large-amplitude solitary waves with $R \equiv 0$ and where F was allowed to vary. While in that paper the local problem near $F = 1$ was one of the main sources of difficulty, in the current paper the local problem (with $F > 1$ fixed) is relatively straightforward. On the other hand, in this paper the global problem is more difficult, primarily because of conditions involving decay at infinity. For solitary waves with constant surface pressure, the height of the free surface decays to its asymptotic value at an exponential rate which depends on the Froude number [Hur08b]. Hur [Hur08b] used these exponential asymptotics together with a moving planes argument to show symmetry and monotonicity under certain assumptions. In [Whe] we removed one of these assumptions, and used the resulting monotonicity in our continuation argument in an essential way. However, with a non-constant surface pressure, the free surface may not decay exponentially (see Appendix A.1 for an example). To avoid placing unnecessary decay conditions on R, we prove monotonicity using a different technique from [Whe]. The main tool is a new maximum principle for small-amplitude waves or, more importantly, large-amplitude waves near infinity (Proposition 2.11). Using this proposition, we are able to show monotonicity directly using simple maximum principle arguments, as was done for the periodic problem [CS04].

Assumptions about decay also come into play when dealing with a compactness condition called (local) *properness* necessary for the definition of a topological degree. In [Whe], we followed Volpert and Volpert [VV03] and obtained local properness by working in weighted Hölder spaces of functions decaying at infinity. These artificial weights then had to be eliminated to get to the final result. In the current paper we avoid weights altogether; we instead obtain properness by simply *restricting the domain* of the nonlinear operator to an appropriate open set (see Lemma 4.12). Using the same technique, one could eliminate all mention of weights from the arguments in [Whe].

In Section 1.3, we use the Dubreil-Jacotin transformation [DJ34] to reformulate (1.1) as a quasi-linear elliptic boundary value problem for a function $w(q,p)$ in a fixed infinite strip. In Section 1.4, we show that, under the extra assumption that the shear flow U is nondecreasing on the bed and concave, alternative (i^-) in Theorem 1.2 implies that the free surface becomes vertical along $\mathscr{C}_2^-$.

In Section 2 we consider the nodal properties of solutions, and give sufficient conditions for solitary waves to be monotone waves of elevation or depression. We will use these properties later in Section 4 to prove (non)compactness (Propositions 4.9 and 4.14) and also to differentiate between the continua $\mathscr{C}_2^\pm, \mathscr{C}_3^\star$ (Proposition 4.11). In Section 2.1, we show that waves with $R \geq 0$ are waves of elevation, while waves with $R \leq 0$ are either waves of depression or have sufficiently large amplitude. In Section 2.2, we investigate the slope η' of the free surface for small-amplitude waves and also for large-amplitude waves near infinity. The main result of this subsection, Proposition 2.11, seems new and of interest even in the irrotational case with $R \equiv 0$. In Section 2.3, we use Proposition 2.11 to show that certain nodal conditions are preserved along continua of solutions.

In Section 3 we focus on regularity and decay properties of (sequences of) solutions, generalizing similar results in [Whe] to the case with $R \not\equiv 0$. In Section 3.1 we show that $\|w\|_{C^{3+\alpha}}$ can be bounded in terms of $\|w_p\|_{L^\infty}$ and $\|R\|_{C^{2+\alpha}}$, where here $w(q,p)$ is the function defined in Section 1.3. In Section 3.2 we introduce the flow force $S(x)$ of a wave, and relate it to the surface pressure $R(x)$. In Section 3.3, we use identities for the flow force together with a translation argument to obtain uniform decay estimates for bounded non-oscillatory sequences (see Definition 1.3).

In Section 4 we prove our main results, Theorems 1.1, 1.2, and 1.4 in the w, q, p variables introduced in Section 1.3. In Section 4.1, we formulate (1.18) as a nonlinear operator equation in a Banach space. We then state several lemmas from [Whe] concerning the associated linearized operators. In Section 4.2, we prove Theorem 1.1 by combining these lemmas with monotonicity and compactness results from Sections 2 and 3. In Section 4.3, we define the continua $\mathscr{C}_2^\pm$ and $\mathscr{C}_3^\star$, and use results from Section 2 to analyze their nodal properties. We then apply a topological degree argument to prove that neither $\mathscr{C}_2^-$ nor $\mathscr{C}_3^\star$ is precompact. Finally, we use this non-compactness in Section 4.4 to prove Theorems 1.2 and 1.4.

In Appendix A.1, we study a family of completely explicit solutions of (1.11) with constant vorticity. This family includes monotone waves of elevation arbitrarily close to stagnation, as well as sequences of monotone waves of depression where the surface pressure $R(0)$ at the trough tends to $-\infty$. In Appendix A.2, we repeat a summary [CS04, Whe] of the key features of the Healey–Simpson degree [HS98] for the reader's convenience. In Appendix A.3, we give several slight variations of standard facts about elliptic problems in infinite strips, mostly without proof.

1.3 Reformulation

Let $\Omega \subset \mathbb{R}^n$ be a domain, possibly unbounded. We say that $\varphi \in C_c^\infty(\Omega)$ if $\varphi \in C^\infty(\Omega)$ and the support of φ is a compact subset of Ω. Similarly $\varphi \in C_c^\infty(\overline{\Omega})$ if $\varphi \in C^\infty(\overline{\Omega})$ and the support of φ is a compact subset of $\overline{\Omega}$. Let k be a nonnegative integer and $\alpha \in [0,1)$. We say that $u \in C^{k+\alpha}(\Omega)$ if $\|\varphi u\|_{C^{k+\alpha}(\Omega)} < \infty$ for all $\varphi \in C_c^\infty(\Omega)$, and similarly $u \in C^{k+\alpha}(\overline{\Omega})$ if $\|\varphi u\|_{C^{k+\alpha}(\Omega)} < \infty$ for all $\varphi \in C_c^\infty(\overline{\Omega})$. If $\|u\|_{C^{k+\alpha}(\Omega)} < \infty$, we say $u \in C_b^{k+\alpha}(\overline{\Omega})$ (the suprema in the definition of the norm range over all of Ω). We say that $u_n \to u$ in $C_{loc}^{k+\alpha}(\overline{\Omega})$ if $\|\varphi(u_n - u)\|_{C^{k+\alpha}(\Omega)} \to 0$ for all $\varphi \in C_c^\infty(\overline{\Omega})$. Finally, we define

$$C_0^{k+\alpha}(\overline{\Omega}) = \left\{ u \in C_b^{k+\alpha}(\overline{\Omega}) : \lim_{r\to\infty} \sup_{|x|=r} |D^\ell u(x)| = 0 \quad \text{for } 0 \le \ell \le k \right\}$$

which is a Banach space under the $C_b^{k+\alpha}(\overline{\Omega})$ norm.

Let $U(y)$ be the asymptotic shear flow fixed in Section 1.2 and let d be the asymptotic depth. We define the Bernoulli constant λ, "total head" Q, and flux m in terms $U(y)$ and d by

$$\lambda = (c - U(0))^2, \qquad Q = \frac{\lambda}{2} + gd, \qquad m = \int_{-d}^{0} (c - U(y))\, dy. \tag{1.6}$$

Since $U < c$, the flux m is positive. By incompressibility ($u_x + v_y = 0$) there exists a stream function ψ defined up to an additive constant by

$$\psi_x = -v, \qquad \psi_y = u - c.$$

The kinematic boundary conditions (1.1b) imply that ψ is constant on the bottom $y = 0$ and also on the free surface $y = \eta(x)$. In particular, the difference

$$\psi(x, -d) - \psi(x, \eta(x)) = -\int_{-d}^{\eta(x)} \psi_y(x,y)\, dy = \int_{-d}^{\eta(x)} (c - u(x,y))\, dy$$

is independent of x. Sending $x \to \infty$ we deduce that this difference is the flux m; we normalize ψ so that $\psi = 0$ on the free surface and $\psi = m$ on the bottom.

Thanks to our no-stagnation assumption that

$$u - c = \psi_y < 0 \tag{1.7}$$

throughout the fluid, the vorticity ω satisfies

$$\omega = v_x - u_y = -\Delta\psi = \gamma(\psi)$$

for some function γ called the *vorticity function* [CS04]. From the asymptotic conditions (1.1d), we see that γ is defined in terms of U via

$$\gamma(\Psi(y)) = -U_y(y), \qquad \text{where } \Psi(y) = \int_0^y (U(y') - c)\, dy'. \tag{1.8}$$

Squaring both sides and integrating, we obtain

$$c - U(y) = \sqrt{\lambda + 2\Gamma(-\Psi(y))}, \tag{1.9}$$

where Γ is the antiderivative

$$\Gamma(p) = \int_0^p \gamma(-s)\, ds$$

of γ.

The Euler equations imply that

$$E = \tfrac{1}{2}|\nabla\psi|^2 + gy + P - \Gamma(-\psi)$$

is constant throughout the fluid. In particular, evaluating E at $x = \pm\infty$ and $y = 0$, we have

$$\tfrac{1}{2}|\nabla\psi|^2 + gy + P - \Gamma(-\psi) \equiv \tfrac{1}{2}\lambda^2 + gd + P_{\text{atm}}. \tag{1.10}$$

Putting this all together and recalling that $P = P_{\text{atm}} + R(x)$ on $y = \eta(x)$, we find that ψ solves

$$\Delta\psi = -\gamma(\psi) \qquad \text{in } -d < y < \eta(x), \tag{1.11a}$$
$$\psi = m \qquad \text{on } y = -d, \tag{1.11b}$$
$$\psi = 0 \qquad \text{on } y = \eta(x), \tag{1.11c}$$
$$\tfrac{1}{2}|\nabla\psi|^2 + g\eta = \tfrac{\lambda}{2} - R \qquad \text{on } y = \eta(x), \tag{1.11d}$$

together with the asymptotic conditions

$$\eta \to 0, \quad \psi_x \to 0, \quad \psi_y \to U(y) - c, \qquad \text{as } x \to \pm\infty, \tag{1.11e}$$

uniformly in y.

We now perform the Dubreil-Jacotin transformation [DJ34]. Since $\psi_y < 0$ throughout the fluid by (1.7), we can use

$$q = x, \qquad p = -\psi,$$

as independent variables. This transforms the fluid domain D_η into the fixed infinite strip

$$\Omega := \{(q,p) \in \mathbb{R}^2 : -m < p < 0\}.$$

The top boundary $T = \{p = 0\}$ of Ω corresponds to the free surface $\{y = \eta(x)\}$, and the bottom boundary $B = \{p = -m\}$ corresponds to the horizontal bed $\{y = 0\}$. Choosing the height $h(q,p) = y + d$ above the bed as the new dependent variable, we can recover the velocity field (u, v) and free surface η via

$$c - u(q, h(q,p)) = \frac{1}{h_p(q,p)}, \qquad v(q, h(q,p)) = -\frac{h_q(q,p)}{h_p(q,p)}, \qquad \eta(q) = h(q,0) - d. \tag{1.12}$$

We call h the *height function*. Note that our assumption that $u < c$ implies $h_p > 0$ and

$$\sup_{D_\eta}(u - c) = -\|h_p\|^{-1}_{L^\infty(\Omega)}.$$

The height function h solves [CS04, CS11]

$$\Big(-\frac{1+h_q^2}{2h_p^2} + \Gamma\Big)_p + \Big(\frac{h_q}{h_p}\Big)_q = 0 \qquad \text{in } \Omega, \tag{1.13a}$$
$$\frac{1+h_q^2}{2h_p^2} + gh = \frac{\lambda}{2} + gd - R \qquad \text{on } T, \tag{1.13b}$$
$$h = 0 \qquad \text{on } B, \tag{1.13c}$$

together with the asymptotic condition

$$\lim_{q\to\pm\infty} h(p,q) = H(p) := \int_{-m}^{p} \frac{ds}{\sqrt{\lambda + 2\Gamma(s)}}, \qquad \lim_{q\to\pm\infty} Dh(p,q) = DH(p), \tag{1.13d}$$

uniformly in p. Here $h(q,p) = H(p)$ corresponds the laminar flow $(u, v, \eta) = (U, 0, 0)$, as can be seen using (1.9) and (1.12). We observe that the regularity $U \in C^{3+\alpha}[-d, 0]$ implies

$$\gamma \in C^{2+\alpha}[0, m], \qquad \Gamma \in C^{3+\alpha}[-m, 0], \qquad H \in C^{4+\alpha}[-m, 0].$$

See [CS04] for the details of the equivalence between (1.13) and (1.1); they only deal with $R \equiv 0$, but the arguments remain unchanged for $R \not\equiv 0$.

Next we consider the Froude number F. From

$$\frac{1}{F^2} = g\int_{-d}^{0} \frac{dy}{(c-U(y))^2} = g\int_{-m}^{0} H_p^3(p)\,dp = g\int_{-m}^{0} \frac{dp}{(\lambda+2\Gamma(p))^{3/2}} \tag{1.14}$$

we see that our assumption $F > 1$ is equivalent to $\lambda > \lambda_{\mathrm{cr}}$, where λ_{cr} is the solution of

$$g\int_{-m}^{0} \frac{dp}{(\lambda_{\mathrm{cr}}+2\Gamma(p))^{3/2}} = 1. \tag{1.15}$$

Such a λ_{cr} exists and is unique since the right hand side of (1.14) is decreasing in λ, tends to 0 as $\lambda \to \infty$, and tends to ∞ as $\lambda \to -2\min_{-m<p<0}\Gamma(p)$. Expressing $d = H(0)$ in terms of m, γ, λ,

$$d = \int_{-m}^{0} \frac{dp}{\sqrt{\lambda+2\Gamma(p)}}, \tag{1.16}$$

we see that $F > 1$ is also equivalent to $d < d_{\mathrm{cr}}$, where d_{cr} is defined in terms of λ_{cr} by

$$d_{\mathrm{cr}} = \int_{-m}^{0} \frac{dp}{\sqrt{\lambda_{\mathrm{cr}}+2\Gamma(p)}}. \tag{1.17}$$

We ultimately formulate (1.1) in terms of the difference

$$w := h - H$$

between the height function $h(q,p)$ and its asymptotic value $H(p)$ at $q = \pm\infty$. Note that $\eta(q) = w(q,0)$. Rewriting (1.13a)–(1.13c) in terms of w, we have

$$\Big(-\frac{1+w_q^2}{2(H_p+w_p)^2}+\Gamma\Big)_p + \Big(\frac{w_q}{H_p+w_p}\Big)_q = 0 \qquad \text{in } \Omega, \tag{1.18a}$$

$$\frac{1+w_q^2}{2(H_p+w_p)^2} + gw = \frac{\lambda}{2} - R \qquad \text{on } T, \tag{1.18b}$$

$$w = 0 \qquad \text{on } B. \tag{1.18c}$$

The assumptions $\inf_{D_\eta} u > -\infty$ and $\sup_{D_\eta} u < c$ in the original variables are equivalent to

$$\inf_{\Omega}(H_p + w_p) > 0, \tag{1.18d}$$

and we enforce the asymptotic condition (1.13d) by requiring

$$w \in C_{\mathrm{b}}^{3+\alpha}(\overline{\Omega}) \cap C_0^2(\overline{\Omega}). \tag{1.18e}$$

We will also usually assume

$$w \text{ is even in } q, \tag{1.18f}$$

in which case w represents a symmetric wave.

We emphasize that here, as in the rest of the paper, the asymptotic shear flow U and depth d are held fixed. Thus $\gamma, \Gamma, \lambda, m, H, Q, F$ are all also fixed. In Sections 2.1, 3.2, and 3.3, we will need to make reference to other asymptotic shear flows with the same vorticity function γ and flux m, but different Bernoulli constants, depths, and heads. To avoid confusion, we will always denote the Bernoulli constants, depths, and heads of these flows by $\widehat{\lambda}$, $\widehat{d}$, and $\widehat{Q}$.

1.4 The case of concave asymptotic shear

As mentioned in Section 1.2, we expect that alternative (i$^-$) in Theorem 1.2, that waves in $\mathscr{C}_2^-$ approach stagnation, can be eliminated, perhaps under additional assumptions on the asymptotic shear flow U. This expectation is based on a comparison with numerical results [AVB94], and also on the simple observation that if the free surface η were held constant, then a negative surface pressure R in (1.11d) would act to increase the relative speed $\sqrt{(c-u)^2+v^2}$ on the free surface.

In this section, we consider the case where the shear flow U is nondecreasing at the bed and concave. Under this extra assumption, we show that, if alternative (i$^-$) of Theorem 1.2 holds, then the free surface η must also become vertical along $\mathscr{C}_2^-$.

Proposition 1.5. *Consider the same situation as in Theorem 1.2, and suppose that the asymptotic shear flow U satisfies $U''(y) \le 0$ for $-d < y < 0$ and $U'(-d) \ge 0$. If alternative* (i$^-$) *holds for $\mathscr{C}_2^-$, then*

$$\sup_{\mathscr{C}_2^-} \|\eta'\|_{L^\infty} = \infty.$$

Note that Proposition 1.5 applies to the irrotational case $\gamma \equiv 0$ as well as the case of constant vorticity $\gamma < 0$. Since by Theorem 1.2 we have $R \le 0$ and $\eta \le 0$ along $\mathscr{C}_2^-$, Proposition 1.5 is an immediate consequence of the following lemma, which is proved using the maximum principle and Hopf lemma in the physical variables (x, y).

Lemma 1.6. *Let u, v, η solve* (1.1) *with $\sup_{D_\eta} u < c$, and suppose that $R + g\eta \le 0$. If the asymptotic shear flow U satisfies $U''(y) \le 0$ for $-d < y < 0$ and $U'(-d) \ge 0$, then*

$$\sup_{D_\eta}(u-c) \le \max\left\{ \sup_{-d<y<0}(U(y)-c), \frac{U(0)-c}{(1+\|\eta'\|_{L^\infty}^2)^{1/2}} \right\}. \tag{1.19}$$

Proof. Defining the vorticity function $\gamma(\psi)$ as in Section 1.3, our assumptions on U imply $\gamma' \le 0$ and $\gamma(m) \le 0$. Differentiating (1.11a) with respect to x, we see that $u - c$ satisfies the elliptic equation

$$\Delta(u-c) = -\gamma'(\psi)(u-c)$$

to which the maximum principle applies. On the bed $y = -d$ we have

$$u_y = v_x - \gamma(m) = -\gamma(m) \ge 0,$$

so u cannot achieve its maximum there by the Hopf lemma. On the free surface, we have

$$\frac{\lambda}{2} = \frac{(u-c)^2+v^2}{2} + R + g\eta = \frac{1+(\eta')^2}{2}(u-c)^2 + R + g\eta.$$

Rearranging and using $R + g\eta \le 0$, we obtain

$$c-u = \left(\frac{\lambda - 2R - 2g\eta}{1+(\eta')^2}\right)^{1/2} \ge \left(\frac{\lambda}{1+\|\eta'\|_{L^\infty}^2}\right)^{1/2} \ge \frac{c-U(0)}{(1+\|\eta'\|_{L^\infty}^2)^{1/2}} \qquad \text{on } y = \eta(x).$$

The statement then follows by applying the maximum principle Lemma A.14. $\square$

2 Elevation, depression, and monotonicity

In this section we will consider nodal properties of solitary waves. We will later use these properties in Section 4.3 to distinguish the continua $\mathscr{C}_2^{\pm}, \mathscr{C}_3^{\star}$ appearing in Theorems 1.2 and 1.4 (Proposition 4.11). This is typical for global continuation/bifurcation problems [Kie04]. More importantly, however, as in [Whe] (but unlike in the periodic case) we will use nodal properties again in Section 4.3 when dealing with (non)compactness in Proposition 4.14 (also see Proposition 4.9). In Section 2.1, we will show that the possible heights of crests and troughs are restricted by the sign of the surface pressure R: waves with $R \geq 0$ are waves of elevation, while waves with $R \leq 0$ are either waves of depression or have a sufficiently tall crest. With $R \equiv 0$, these reduce to results contained in [Whe]. The main result of Section 2.2, Proposition 2.11, asserts that small-amplitude waves inherit the monotonicity of their surface pressures R, and seems new and of interest even in the irrotational case with $R \equiv 0$. Proposition 2.11 can also be applied to large-amplitude waves near $x = \pm\infty$, which we will do in Section 2.3 to show that certain nodal properties are both open an closed conditions in $C^3_{\mathrm{b}}(\overline{\Omega}) \cap C^1_0(\overline{\Omega})$. Throughout this section our basic tools will be the maximum principles Lemma A.14 and Lemma A.15.

Because it is more natural, we will state the results in this section in terms of an arbitrary surface pressure $R(q)$ rather than a 1-parameter family $R(q;\beta)$.

2.1 Crests and troughs

We now introduce a one-parameter family of shear flows with the same flux m and vorticity function γ as U, but with different Bernoulli constants $\widehat{\lambda}$, asymptotic depths $\widehat{d}$, and total heads $\widehat{Q}$. The corresponding height functions are

$$\widehat{H}(p) = \widehat{H}(p;\widehat{\lambda}) = \int_{-m}^{p} \frac{ds}{(\widehat{\lambda} + 2\Gamma(s))^{1/2}}, \tag{2.1}$$

with $\widehat{d} = \widehat{H}(0;\widehat{\lambda})$. Setting $\Gamma_{\min} = \min_{-m<p<0} \Gamma$, $\widehat{H}$ is well-defined for $\widehat{\lambda} > -2\Gamma_{\min}$ and possibly also for $\widehat{\lambda} = -2\Gamma_{\min}$. The results in this section are based on the following lemma:

Lemma 2.1. *For the shear flows with height function $\widehat{H}(p) = \widehat{H}(p;\widehat{\lambda})$ given by* (2.1)*,*

(a) *The Bernoulli constant $\widehat{\lambda}$ is a strictly decreasing and strictly convex function $\widehat{\lambda} = \widetilde{\lambda}(\widehat{d})$ of the asymptotic depth $\widehat{d} \in (0, d_{\mathrm{m}})$, with $\widehat{\lambda} \to \infty$ as $\widehat{d} \to 0$ and $\widehat{\lambda} \to -2\Gamma_{\min}$ as $\widehat{d} \to d_{\mathrm{m}}$. Here*

$$d_{\mathrm{m}} := \lim_{\widehat{\lambda} \downarrow -2\Gamma_{\min}} \int_{-m}^{0} \frac{dp}{(\widehat{\lambda} + 2\Gamma(p))^{1/2}} \leq +\infty. \tag{2.2}$$

(b) *The total head $\widehat{Q}$ is a strictly convex function $\widehat{Q} = \widetilde{Q}(\widehat{d})$ of the asymptotic depth $\widehat{d} \in (0, d_{\mathrm{m}})$, with a unique minimum at $\widehat{d} = d_{\mathrm{cr}}$.*

Proof. Expressing $\widehat{d}$ as a function $\widetilde{d}(\lambda)$,

$$\widehat{d} = \widetilde{d}(\lambda) := \widehat{H}(0;\widehat{\lambda}) = \int_{-m}^{0} \frac{dp}{(\widehat{\lambda} + 2\Gamma(p))^{1/2}},$$

we have by definition that $\widetilde{d}(\widehat{\lambda}) \to d_{\mathrm{m}}$ as $\widehat{\lambda} \to -2\Gamma_{\min}$. Differentiating under the integral, we easily check that $\widetilde{d}$ is a strictly decreasing and strictly convex function of $\widehat{\lambda} \in (-2\Gamma_{\min}, \infty)$. Thus it has a unique inverse $\widehat{\lambda} = \widetilde{\lambda}(\widehat{d})$, defined for $\widehat{d} \in (0, d_{\mathrm{m}})$, which is also strictly decreasing and strictly convex. Indeed, we compute

$$\widetilde{\lambda}'(\widehat{d}) = -2\bigg(\int_{-m}^{0} \frac{ds}{\big(\widetilde{\lambda}(\widehat{d}) + 2\Gamma(s)\big)^{3/2}} \bigg)^{-1}. \tag{2.3}$$

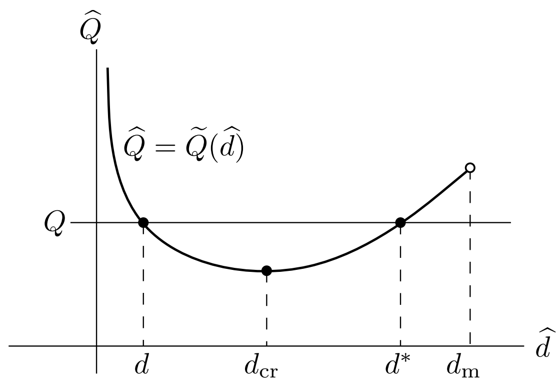

Figure 2: The curve $\widehat{Q} = \widetilde{Q}(\widehat{d})$ and depths $0 < d < d_{\rm cr} < d^* \le d_{\rm m} \le \infty$.

Writing

$$\widehat{Q} = \widetilde{Q}(\widehat{d}) := \tfrac{1}{2}\widetilde{\lambda}(\widehat{d}) + g\widehat{d},$$

we therefore also have that $\widetilde{Q}$ is a strictly convex function of $\widehat{d} \in (0, d_{\rm m})$. Differentiating, we find

$$\widetilde{Q}'(\widehat{d}) = \frac{1}{2}\widetilde{\lambda}'(\widehat{d}) + g = -\left(\int_{-m}^{0} \frac{ds}{(\widetilde{\lambda}(\widehat{d}) + 2\Gamma(s))^{3/2}}\right)^{-1} + g$$

and hence that $\widetilde{Q}$ achieves its unique minimum at $\widehat{d} = d_{\rm cr}$, where $d_{\rm cr}$ was defined in (1.17). □

Since our assumption that $F > 1$ is equivalent to $d < d_{\rm cr}$, the convexity of $\widetilde{Q}$ implies that there exists at most one d^* with $d_{\rm cr} < d^* < d_{\rm m}$ and

$$\widetilde{Q}(d^*) = \widetilde{Q}(d). \tag{2.4}$$

This allows us to make the following definition:

Definition 2.2 (The depth d^*)**.** If (2.4) has a (necessarily unique) solution, we denote it by d^*. Otherwise we set $d^* = d_{\rm m}$.

See Figure 2 for a graphical depiction of the depths $d, d^*, d_{\rm cr}, d_{\rm m}$. In the case of constant vorticity γ, $\widetilde{Q}$ and $d_{\rm m}$ can be computed explicitly,

$$\widetilde{Q}(\widehat{d}) = \frac{1}{2}\left(\frac{m}{\widehat{d}} - \frac{\widehat{d}\gamma}{2}\right)^2 + g\widehat{d}, \qquad d_{\rm m} = \left|\frac{2m}{\gamma}\right|^{1/2}, \tag{2.5}$$

where $d_{\rm m} = +\infty$ in the irrotational case $\gamma \equiv 0$.

We note that the following two lemmas remain true (with identical proofs) if we drop our usual assumption that the Froude number is supercritical.

Lemma 2.3 (Pressure at a trough)**.** *Let $w \in C^2_{\rm b}(\overline{\Omega}) \cap C^0_0(\overline{\Omega})$ be a nontrivial solution of* (1.18a)–(1.18d) *and suppose that the corresponding free surface $\eta(q) = w(q, 0)$ achieves its minimum value $\eta_{\min} = \eta(q_0) \le 0$. Then*

$$R(q_0) < \widetilde{Q}(d) - \widetilde{Q}(d + \eta_{\min}). \tag{2.6}$$

Proof. The idea of the proof is to use $\widehat{H}(p; \widehat{\lambda})$ as a comparison function and then apply the Hopf lemma. For convenience set $h = H + w$, and let

$$\widehat{d} = d + \eta_{\min} = h(q_0, 0)$$

be the depth of the fluid at q_0. From (1.18d) we have $\inf_\Omega h_p > 0$ and hence $\widehat{d} > 0$, and on the other hand $\eta_{\min} \le 0$ implies $\widehat{d} \le d < d_{\mathrm{m}}$. By Lemma 2.1(a), there therefore exists $\widehat{\lambda} \ge \lambda$ such that $H(0;\widehat{\lambda}) = \widehat{d}$. Abbreviating $\widehat{H}(p) = \widehat{H}(p;\widehat{\lambda})$, we set

$$\varphi = h - \widehat{H}.$$

A direct computation shows that φ satisfies

$$(1+h_q^2)\varphi_{pp} - 2h_p h_q \varphi_{pq} + h_p^2 \varphi_{qq} + b_1\varphi_q + b_2\varphi_p = 0 \tag{2.7}$$

in Ω, where

$$b_1 = \gamma(3\widehat{H}_p^2 + 3\widehat{H}_p\varphi_p + \varphi_p^2), \qquad b_2 = -\gamma \widehat{H}_p^3 \varphi_q.$$

We observe that (2.7) is a uniformly elliptic equation for φ; indeed the coefficients satisfy

$$(1+h_q^2)h_p^2 - (h_p h_q)^2 = h_p^2 > 0$$

uniformly in Ω. Since $\widehat{\lambda} \ge \lambda$, the formulas for $\widehat{H}(p)$ and $H(p)$ also give

$$\lim_{|q|\to\infty} \varphi(q,p) = H(p) - \widehat{H}(p) \ge 0,$$

uniformly in p. Since $\varphi = 0$ on B and $\varphi = \eta - \eta_{\min} \ge 0$ on T, the maximum principle Lemma A.14 therefore implies $\varphi > 0$ in Ω. Applying the Hopf lemma at $(q_0, 0)$ then yields

$$\varphi_p(q_0,0) = h_p(q_0,0) - \frac{1}{\sqrt{\widehat{\lambda}}} < 0.$$

Combining this with the boundary condition (1.13b) we have

$$\frac{\widehat{\lambda}}{2} + g(\widehat{d} - d) < \frac{1}{2h_p(q_0,0)^2} + g(h(q_0,0) - d) = \frac{\lambda}{2} - R(q_0),$$

and hence

$$R(q_0) < (\tfrac{1}{2}\lambda + gd) - (\tfrac{1}{2}\widehat{\lambda} + g\widehat{d}) = \widetilde{Q}(d) - \widetilde{Q}(d + \eta_{\min})$$

as desired. ☐

Lemma 2.4 (Pressure at a crest). *Let $w \in C_{\mathrm{b}}^2(\overline{\Omega}) \cap C_0^0(\overline{\Omega})$ be a nontrivial solution of* (1.18a)–(1.18d) *and suppose that the corresponding free surface $\eta(q) = w(q,0)$ achieves its maximum value $\eta_{\max} = \eta(q_0) < d_{\mathrm{m}} - d$. Then*

$$R(q_0) > \widetilde{Q}(d) - \widetilde{Q}(d + \eta_{\max}). \tag{2.8}$$

Proof. We use the same argument as in the proof of Lemma 2.3. For convenience set $h = H + w$, and let

$$\widehat{d} = d + \eta_{\max} = h(q_0, 0)$$

be the depth of the fluid at q_0. By assumption $d \le \widehat{d} < d_{\mathrm{m}}$, so by Lemma 2.1(a) there exists $-2\Gamma_{\min} < \widehat{\lambda} \le \lambda$ such that $\widehat{H}(0;\widehat{\lambda}) = \widehat{d}$. Abbreviating $\widehat{H}(p) = \widehat{H}(p;\widehat{\lambda})$, we have as in the proof of Lemma 2.3 that $\varphi := h - \widehat{H}$ solves the elliptic equation (2.7) in Ω. Since $\varphi = 0$ on B, $\varphi = \eta - \eta_{\max} \le 0$ on T, and $\lim_{|q|\to\infty} \varphi(q,p) \le 0$, Lemma A.14 implies $\varphi < 0$ in Ω. Applying the Hopf lemma at $(q_0, 0)$ then yields

$$\varphi_p(q_0,0) = h_p(q_0,0) - \frac{1}{\sqrt{\widehat{\lambda}}} > 0.$$

Combining this with the boundary condition and rearranging, we have

$$R(q_0) < (\tfrac{1}{2}\widehat{\lambda} + g\widehat{d}) - (\tfrac{1}{2}\lambda + gd) = \widetilde{Q}(d) - \widetilde{Q}(d + \eta_{\min})$$

as desired. □

Corollary 2.5 (Elevation)**.** *All nontrivial (supercritical) solitary waves with $R \geq 0$ are waves of elevation. More precisely, if w solves* (1.18a)–(1.18d) *with $R \geq 0$, then $w > 0$ in $\Omega \cup T$ unless $w \equiv 0$.*

Proof. Let $w \not\equiv 0$ solve (1.18a)–(1.18d) with $R \geq 0$, and suppose for contradiction that $\eta(q) = w(q,0)$ achieves its minimum $\eta_{\min} = \eta(q_0) \leq 0$. By Lemma 2.3 and the nonnegativity of R, we have

$$0 \leq R(q_0) < \widetilde{Q}(d) - \widetilde{Q}(d + \eta_{\min}).$$

Now by supercriticality $d < d_{\mathrm{cr}}$, so in particular $d + \eta_{\min} < d_{\mathrm{cr}}$. But by Lemma 2.1(b), $\widetilde{Q}(\widehat{d})$ is decreasing for $0 < \widehat{d} < d_{\mathrm{cr}}$, so $\widetilde{Q}(d) \leq \widetilde{Q}(d + \eta_{\min})$, a contradiction. Thus $w > 0$ on T. Since $\varphi = w \in C^2_{\mathrm{b}}(\overline{\Omega}) \cap C^0_0(\overline{\Omega})$ also vanishes on B and satisfies the uniformly elliptic equation (2.7) with $\widehat{H} = H$, the maximum principle Lemma A.14 then implies $w > 0$ in Ω. □

Corollary 2.6 (Depression)**.** *All nontrivial (supercritical) solitary waves with $R \leq 0$ and $d + \sup_{\mathbb{R}} \eta < d^*$ are waves of depression. More precisely, suppose w solves* (1.18a)–(1.18d) *with $R \leq 0$. If $\sup_T w < d^* - d$, then $w < 0$ on $\Omega \cup T$ unless $w \equiv 0$.*

Proof. Let $w \not\equiv 0$ solve (1.18a)–(1.18d) with $R \leq 0$, and suppose for contradiction that $\eta(q) = w(q,0)$ achieves its maximum $\eta_{\max} = \eta(q_0) < d^* - d$. By Lemma 2.4 and the nonnegativity of R, we have

$$0 \geq R(q_0) > \widetilde{Q}(d) - \widetilde{Q}(d + \eta_{\max}). \tag{2.9}$$

Now by assumption $d \leq d + \eta_{\max} < d^*$, and the convexity of $\widetilde{Q}$ (Lemma 2.1(b)) and definition of d^* imply that $\widetilde{Q}(\widehat{d}) \leq \widetilde{Q}(d)$ for $d < \widehat{d} < d^*$. Thus $\widetilde{Q}(d + \eta_{\max}) \leq \widetilde{Q}(d)$, contradicting (2.9). Therefore $w < 0$ on T. Since $\varphi = w \in C^2_{\mathrm{b}}(\overline{\Omega}) \cap C^0_0(\overline{\Omega})$ also vanishes on B and satisfies the uniformly elliptic equation (2.7) with $\widehat{H} = H$, the maximum principle Lemma A.14 implies $w < 0$ in Ω. □

2.2 Monotonicity

In this section we will prove several facts about the relationship between the sign of w_q (which determines the sign of the slope η' of the free surface) and the pressure disturbance R. The most important of these is Proposition 2.11. The idea in all of the proofs is to write $w = v/\Phi$ for some function $\Phi(p)$ and then show that v satisfies a linear equation to which the Hopf-type maximum principle Lemma A.15 applies.

We being by considering the linear problem

$$\left(\frac{w_p}{H_p^3}\right)_p + \left(\frac{w_q}{H_p}\right)_q = 0 \qquad \text{in } \Omega, \tag{2.10a}$$

$$\frac{w_p}{H_p^3} - gw = R \qquad \text{on } T, \tag{2.10b}$$

$$w = 0 \qquad \text{on } B, \tag{2.10c}$$

obtained by linearizing (1.18a)–(1.18c) about $w = 0$. A version of the following lemma was proven by Craig and Sternberg [CS88] in the irrotational case using Green's functions and the calculus of residues. Here we give a different proof in our more general setting which has the virtue of depending only on the strong maximum principle and the Hopf lemma.

Lemma 2.7. *Suppose that $w \in C^2_{\mathrm{b}}(\overline{\Omega}) \cap C^0_0(\overline{\Omega})$ solves the linearized equation* (2.10) *with $R \geq 0$. Then $w > 0$ in $\Omega \cup T$ unless $w \equiv 0$.*

Proof. Because of the signs of the coefficients in (2.10b), we cannot apply Lemma A.15 directly to (2.10). Instead we will make the change of dependent variable

$$v(q,p) = \frac{w(q,p)}{\Phi(p)}, \qquad \Phi(p) = \int_{-m}^{p} H_p^3(s)\, ds + \varepsilon,$$

where $\varepsilon > 0$ is a constant to be determined. Since Φ is uniformly bounded away from 0, we easily check that $v \in C_{\mathrm{b}}^2(\overline{\Omega}) \cap C_0^0(\overline{\Omega})$, and that it suffices to show $v > 0$ in $\Omega \cup T$. Looking at (1.14), we see that $F > 1$ implies $g \int_{-m}^{0} H_p^3\, dp < 1$, so we can fix ε small enough that

$$g\Phi(0) < 1. \tag{2.11}$$

On T we then calculate

$$R = \frac{w_p}{H_p^3} - gw = \frac{\Phi}{H_p^3} v_p + (1 - g\Phi)v, \tag{2.12}$$

where the coefficients in front of v_p and v in (2.12) are now both positive thanks to (2.11). Since v vanishes on B and satisfies the uniformly elliptic equation

$$\Big(\frac{\Phi}{H_p^3} v_p\Big)_p + \Big(\frac{\Phi}{H_p} v_q\Big)_q + v_p = 0 \tag{2.13}$$

in Ω, Lemma A.15 implies $v > 0$ in $\Omega \cup T$, and hence $w > 0$ in $\Omega \cup T$. □

Our proof of Lemma 2.7 extends to small-amplitude solutions of the full nonlinear equation (1.18). To perform the extension, we first make a slightly more complicated choice of the function $\Phi(p)$:

Lemma 2.8. *There exists $\varepsilon > 0$ such that the function*

$$\Phi(p) = \Big(\varepsilon + \int_{-m}^{p} H_p^3(s)\, ds\Big)^{1-\varepsilon}$$

satisfies

$$\Big(\frac{\Phi_p}{H_p^3}\Big)_p < 0 \text{ on } [-m,0], \qquad \Phi > 0 \text{ on } [-m,0], \qquad \Big(\frac{\Phi_p}{H_p^3} - g\Phi\Big)(0) > 0. \tag{2.14}$$

Proof. That $\Phi > 0$ on $[-m,0]$ for any $\varepsilon > 0$ is clear from the definition. Computing

$$\Phi_p = (1-\varepsilon) H_p^3 \Big(\varepsilon + \int_{-m}^{p} H_p^3(s;\lambda)\, ds\Big)^{-\varepsilon}$$

we see that, if $0 < \varepsilon < 1$,

$$\Big(\frac{\Phi_p}{H_p^3}\Big)_p = -\varepsilon(1-\varepsilon) H_p^3 \Big(\varepsilon + \int_{-m}^{p} H_p^3(s)\, ds\Big)^{-1-\varepsilon} < 0 \text{ for } p \in [-m,0].$$

Finally, we observe that

$$\frac{\Phi_p}{H_p^3}(0) - g\Phi(0) = (1-\varepsilon)\Big(\varepsilon + \int_{-m}^{0} H_p^3(s)\, ds\Big)^{-\varepsilon} - g\Big(\varepsilon + \int_{-m}^{0} H_p^3(s)\, ds\Big)^{1-\varepsilon}$$

depends continuously on ε near 0 and is $1 - g\int_{-m}^{0} H_p^3\, dp = 1 - F^{-2} > 0$ when $\varepsilon = 0$. □

The proof of Proposition 2.11 is based on the proof of Lemma 2.9 below. We note that a much stronger version of Lemma 2.9 has already been proved using different methods in Corollaries 2.5 and 2.6.

Lemma 2.9. *There exists $\delta > 0$ such that the following holds. Let $w \in C_{\mathrm{b}}^2(\overline{\Omega}) \cap C_0^0(\overline{\Omega})$ be a solution of* (1.18a)–(1.18d) *with $w \not\equiv 0$ and $\|w\|_{C^1(\Omega)} < \delta$. Then*

(a) *If $R \geq 0$, then $w > 0$ in $\Omega \cup T$.*

(b) *If $R \leq 0$, then $w < 0$ in $\Omega \cup T$.*

Proof. Setting $h = H + w$ for convenience, we first rewrite (1.18a)–(1.18b) in the non-divergence form

$$(1 + h_q^2)w_{pp} - 2h_p h_q w_{pq} + h_p^2 w_{qq} + (1 + h_q^2)H_{pp} + \gamma(H_p + w_p)^3 = 0 \qquad \text{in } \Omega, \tag{2.15}$$

$$-\frac{w_q^2}{2H_p^2} + \frac{(1 + w_q^2)(w_p + 2H_p)}{2H_p^2 h_p^2} w_p - gw = R \qquad \text{on } T. \tag{2.16}$$

Letting $\Phi(p)$ be the function guaranteed by Lemma 2.8, we then define

$$v = \frac{w}{\Phi} \in C_{\mathrm{b}}^2(\overline{\Omega}) \cap C_0^0(\overline{\Omega}),$$

and observe that v vanishes on B. Since $\Phi > 0$, we will be done if we can prove (a) and (b) with w replaced by v. Substituting $w = \Phi v$ in (2.15)–(2.16) we find

$$\Phi(1 + h_q)^2 v_{pp} - 2\Phi h_q h_p v_{qp} + \Phi h_p^2 v_{qq} + bv_p + cv = r_1 \qquad \text{in } \Omega, \tag{2.17}$$

$$dv_p + fv = R + r_2 \qquad \text{on } T, \tag{2.18}$$

where the coefficients b, c, d, f are given by

$$b = 3\gamma H_p^2 \Phi + 2\Phi_p, \qquad c = H_p^3 \Big(\frac{\Phi_p}{H_p^3}\Big)_p, \qquad d = \frac{\Phi}{H_p^3}, \qquad f = \frac{\Phi_p}{H_p^3} - g\Phi, \tag{2.19}$$

and the remainder terms r_1, r_2 are rational expressions satisfying

$$r_i = O(v_q^2 + v_p^2 + v^2) \text{ as } (v_q, v_p, v) \to 0, \qquad i = 1, 2. \tag{2.20}$$

Since Φ is bounded away from zero, (1.18d) guarantees that (2.17) is a uniformly elliptic equation for v. By (2.14), the coefficients satisfy $c < 0$ and $d, f > 0$.

We can also easily regroup terms so that r_1, r_2 are absorbed into the coefficients, transforming (2.17)–(2.18) into

$$\Phi(1 + h_q)^2 v_{pp} - 2\Phi h_q h_p v_{qp} + \Phi h_p^2 v_{qq} + \widetilde{b}_1 v_q + \widetilde{b}_2 v_p + \widetilde{c} v = 0 \qquad \text{in } \Omega, \tag{2.21}$$

$$\widetilde{d}_1 v_q + \widetilde{d}_2 v_p + \widetilde{f} v = R \qquad \text{on } T, \tag{2.22}$$

with $\widetilde{b}_1, \widetilde{b}_2, \widetilde{c} \in C_{\mathrm{b}}^1(\overline{\Omega})$ and $\widetilde{d}_1, \widetilde{d}_2, \widetilde{f} \in C_{\mathrm{b}}^1(T)$. Thanks to the bounds (2.20) on r_1, r_2, for $\|w\|_{C^1(\Omega)}$ sufficiently small, these modified coefficients will still have $\widetilde{c} < 0$ and $\widetilde{d}_2, \widetilde{f} > 0$. Applying Lemma A.15, we then obtain $v > 0$ in $\Omega \cup T$ if $R \geq 0$ and $v < 0$ in $\Omega \cup T$ if $R \leq 0$. □

Differentiating the equations with respect to q, we now obtain analogues of Lemmas 2.7 and 2.9 for w_q. These are most naturally stated in the half-infinite strip

$$\Omega^+ = \{(q, p) : q > 0, \ -m < p < 0\}, \tag{2.23}$$

with top, bottom, and left boundary portions

$$T^+ = \{(q, 0) : q > 0\}, \quad B^+ = \{(q, -m) : q > 0\}, \quad L^+ = \{(0, p) : -m < p < 0\}. \tag{2.24}$$

Lemma 2.10. *Suppose that $w \in C_{\mathrm{b}}^3(\overline{\Omega^+}) \cap C_0^1(\overline{\Omega^+})$ solves the linearized equation* (2.10)*. If $w_q \leq 0$ on L^+ and $R_q \geq 0$ on T^+, then $w_q < 0$ in $\Omega^+ \cup T^+$ unless $w \equiv 0$.*

Proof. Define Φ, v as in the proof of Lemma 2.7, and consider the function

$$u = v_q \in C_{\mathrm{b}}^2(\overline{\Omega^+}) \cap C_0^0(\overline{\Omega^+}).$$

Since $w_q = \Phi u$ and $\Phi > 0$, it is enough to show that u has the appropriate sign on $\Omega^+ \cup T^+$. Because the coefficients in (2.10) are independent of q, u solves the same uniformly elliptic equation (2.13) as v does (though of course only in Ω^+), together with

$$\frac{\Phi}{H_p^3} u_p + (1 - g\Phi)u = R_q \leq 0 \text{ on } T^+ \tag{2.25}$$

and $u = 0$ on B^+. By assumption $w_q \leq 0$ on L^+, so we also have $u \leq 0$ on L^+ and hence $u \leq 0$ on $\partial\Omega^+ \setminus T^+$. As in the proof of Lemma 2.7, the coefficients in front of u_p and u in (2.25) are both positive, so the statement now follows from Lemma A.15. □

We now come to the most important result of this section, which concerns monotonicity properties of solutions to the nonlinear equation.

Proposition 2.11. *There exists $\delta > 0$ such that the following holds. Let $w \in C_{\mathrm{b}}^3(\overline{\Omega^+}) \cap C_0^1(\overline{\Omega^+})$ be a solution of* (1.18a)–(1.18d) *with $w \not\equiv 0$ and $\|w\|_{C^2(\Omega)} < \delta$. Then*

(a) *If $w_q \leq 0$ on L^+ and $R_q \leq 0$ on T^+, then $w_q < 0$ in $\Omega^+ \cup T^+$.*

(b) *If $w_q \geq 0$ on L^+ and $R_q \geq 0$ on T^+, then $w_q > 0$ in $\Omega^+ \cup T^+$.*

Proof. Define Φ, v as in the proof of Lemma 2.9, and set $u = v_q \in C_{\mathrm{b}}^2(\overline{\Omega^+}) \cap C_0^0(\overline{\Omega^+})$. As in the proof of Lemma 2.10, it is enough to show that u has the appropriate sign on $\Omega^+ \cup T^+$. Note that u vanishes on B^+. Differentiating (2.17)–(2.18) yields

$$\Phi(1 + h_q)^2 u_{pp} - 2\Phi h_q h_p u_{qp} + \Phi h_p^2 u_{qq} + b u_p + c u = r_3 \qquad \text{in } \Omega, \tag{2.26}$$

$$d u_p + f u = R_q + r_4 \qquad \text{on } T, \tag{2.27}$$

where $c < 0$ and $d, f > 0$ are given by (2.19) and the remainder terms r_3, r_4 are rational expressions satisfying

$$r_i = O\Big((|v| + |v_q| + |v_p|)(|v_q| + |v_{qq}| + |v_{qp}|)\Big) \text{ as } (v, v_q, v_p, v_{qq}, v_{qp}) \to 0, \qquad i = 3, 4. \tag{2.28}$$

As in the proof of Lemma 2.9, we can absorb r_3, r_4 into the definitions of the coefficients, getting

$$\Phi(1 + h_q)^2 u_{pp} - 2\Phi h_q h_p u_{qp} + \Phi h_p^2 u_{qq} + \widetilde{b}_1 u_q + \widetilde{b}_2 u_p + \widetilde{c} u = 0 \qquad \text{in } \Omega^+, \tag{2.29}$$

$$\widetilde{d}_1 u_q + \widetilde{d}_2 u_p + \widetilde{f} u = R_q \qquad \text{on } T^+, \tag{2.30}$$

where $\widetilde{b}_1, \widetilde{b}_2, \widetilde{c} \in C_{\mathrm{b}}^1(\overline{\Omega^+})$ and $\widetilde{d}_1, \widetilde{d}_2, \widetilde{f} \in C_{\mathrm{b}}^1(\overline{T^+})$. The first equation (2.29) is a uniformly elliptic equation for u. Thanks to (2.28), we also have $\widetilde{c} < 0$ and $\widetilde{d}_2, \widetilde{f} > 0$, provided $\|w\|_{C^2(\Omega)}$ is sufficiently small. Thus Lemma A.15 implies $u > 0$ in $\Omega^+ \cup T^+$ when $R_q \geq 0$ on T^+ and $w_q \geq 0$ on L^+, and similarly $u < 0$ in $\Omega^+ \cup T^+$ when $R_q \leq 0$ on T^+ and $w_q \leq 0$ on L^+. □

In Lemma 2.10 and Proposition 2.11, we have not assumed that w is even in q. Thus their statements remain true if Ω^+ is replaced by any half-strip $(-m, 0) \times (q_0, \infty)$.

2.3 Preservation of nodal properties

In this section we use Proposition 2.11 to address more complicated nodal properties involving second and third partial derivatives of w. We remark that the results of this subsection remain true if we set the Hölder parameter $\alpha = 0$ in (1.18e). For solutions w of (1.18) (which includes the assumption that w is even in q), the *nodal elevation properties* are the following five conditions:

$$w_q < 0 \text{ in } \Omega^+ \cup T^+, \tag{2.31a}$$

$$w_{qq} < 0 \text{ on } L^+, \tag{2.31b}$$

$$w_{qp} < 0 \text{ on } B^+, \tag{2.31c}$$

$$w_{qqp}(0,-m) < 0, \tag{2.31d}$$

$$w_{qq}(0,0) < 0. \tag{2.31e}$$

Here the sets Ω^+, T^+, L^+ were defined in (2.23) and (2.24). The *nodal depression properties* are the same as the nodal elevation properties except that all of the inequalities are reversed:

$$w_q > 0 \text{ in } \Omega^+ \cup T^+, \tag{2.32a}$$

$$w_{qq} > 0 \text{ on } L^+, \tag{2.32b}$$

$$w_{qp} > 0 \text{ on } B^+, \tag{2.32c}$$

$$w_{qqp}(0,-m) > 0, \tag{2.32d}$$

$$w_{qq}(0,0) > 0. \tag{2.32e}$$

Since $w \to 0$ as $q \to \pm\infty$, the nodal elevation properties (2.31) imply $w > 0$ in $\Omega^+ \cup T^+$ and hence that w represents a monotone wave of elevation. Similarly the nodal depression properties (2.32) imply that w represents a monotone wave of depression. As in [CS04, CS07], the nodal properties (2.32) and (2.31) are motivated by the following fact:

Lemma 2.12. *Let w be a solution of* (1.18)*.*

(a) *If $w_q < 0$ in $\Omega^+ \cup T^+$, then the nodal elevation properties* (2.31) *hold.*

(b) *If $w_q > 0$ in $\Omega^+ \cup T^+$, then the nodal depression properties* (2.32) *hold.*

Proof. The proofs of (a) and (b) are the same, so we only prove (a). Differentiating (1.18a) with respect to q we discover that $\varphi = w_q$ satisfies the equation

$$\partial_p\Big(\frac{1+w_q^2}{h_p^3}\partial_p\varphi - \frac{w_q}{h_p^2}\partial_q\varphi\Big) + \partial_q\Big(-\frac{w_q}{h_p^2}\partial_p\varphi + \frac{1}{h_p}\partial_q\varphi\Big) = 0 \tag{2.33}$$

in Ω^+. Since $\inf_\Omega h_p > 0$, we easily check that (2.33) is a uniformly elliptic equation for φ. Since $w_q < 0$ in Ω^+ and $w_q = 0$ on $B^+ \cup L^+$, the Hopf lemma then immediately implies (2.31b) and (2.31c). Since $w_{qp} = w_{qq} = 0$ at $(0,-m)$, (2.31d) follows from Serrin's edge point lemma. It remains to show (2.31e). Since $w_q = w_{pq} = w_{qqq} = 0$ at $(0,0)$, Serrin's edge point lemma implies that either $w_{qq}(0,0) < 0$ or $w_{qq}(0,0) = 0$ and $w_{qpp}(0,0) > 0$. Expanding (2.33), we get

$$\begin{aligned} &\frac{2w_q w_{qq} h_{pp}}{h_p^3} - \frac{3w_{pq}w_q^2 h_{pp}}{h_p^4} - \frac{3w_{pq}h_{pp}}{h_p^4} + \frac{w_{qqq}}{h_p} \\ &\quad - \frac{3w_{pq}w_{qq}}{h_p^2} + \frac{w_{ppq}w_q^2}{h_p^3} - \frac{2w_{pqq}w_q}{h_p^2} + \frac{4w_{pq}^2 w_q}{h_p^3} + \frac{w_{ppq}}{h_p^3} = 0. \end{aligned} \tag{2.34}$$

Evaluating (2.34) at $(0,0)$, $w_q = w_{pq} = w_{qqq} = 0$ implies $w_{qpp} = 0$. Thus the only possibility is $w_{qq}(0,0) < 0$, i.e. that (2.31e) holds. We remark that this last step of the argument differs from [CS07], where instead the boundary condition is differentiated twice (in our case this would introduce R_{qq} terms with indeterminate sign). ☐

Unlike in the periodic case, (2.31) and (2.32) are *not* open conditions in $C_{\mathrm{b}}^3(\overline{\Omega})$ since Ω is unbounded. Nevertheless, we have the following lemma:

Lemma 2.13 (Open condition)**.** *Fix w^* satisfying* (1.18) *and let $M > 0$.*

(a) *If w^* satisfies the nodal elevation conditions* (2.31)*, then there exists $\varepsilon > 0$ such that every solution w of* (1.18) *with $\|w - w^*\|_{C^3(\Omega)} < \varepsilon$ and $R_q \leq 0$ for $q > M$ also satisfies* (2.31)*.*

(b) *If w^* satisfies the nodal depression conditions* (2.32)*, then there exists $\varepsilon > 0$ such that every solution w of* (1.18) *with $\|w - w^*\|_{C^3(\Omega)} < \varepsilon$ and $R_q \geq 0$ for $q > M$ also satisfies* (2.32)*.*

Proof. The proofs of (a) and (b) are the same, so we only prove (a). Let $M > 0$ and fix w^* as in the statement of the lemma. By Lemma 2.12, it is enough to show that $w_q < 0$ in $\Omega^+ \cup T^+$ if $\|w - w^*\|_{C^3(\Omega)}$ is sufficiently small. Letting $K > M$ be a constant to be determined, we first we split Ω^+ into two overlapping regions,

$$\Omega_1 = \{(q,p) \in \Omega^+ : 0 < q < 2K\}, \qquad \Omega_2 = \{(q,p) \in \Omega^+ : q > K\},$$

and set $B_i = B^+ \cap \partial\Omega_i$ and $T_i = T^+ \cap \partial\Omega_i$. On the bounded set Ω_1, we can argue as in [CS07]: Taylor expanding near the sides L^+, B_1, T_1 and the corners $(0,0)$,$(0,-m)$, the nodal conditions (2.31) for w^* imply that $w_q < 0$ in $\Omega_1 \cup T_1$ whenever $\|w - w^*\|_{C^3(\Omega)} < \varepsilon_K$ for some $\varepsilon_K > 0$ depending on K.

Next we consider the unbounded set Ω_2. Letting $\delta > 0$ be as in Proposition 2.11, the assumption $w^* \in C^2_0(\overline{\Omega})$ allows us to fix $K > M$ large enough that $\|w^*\|_{C^2(\Omega_2)} < \delta/2$. Assuming that $\|w - w^*\|_{C^3(\Omega)} < \varepsilon := \min(\frac{\delta}{2}, \varepsilon_K)$, our above argument then guarantees $w_q < 0$ in $\Omega_1 \cup T_1$. In particular, $w_q \leq 0$ on the left boundary $\{q = K\}$ of Ω_2. Since $\|w\|_{C^2(\Omega_2)} < \delta$ and $R_q \leq 0$ on T_2, Proposition 2.11 implies $w_q < 0$ in $\Omega_2 \cup T_2$. $\square$

As in [CS04], the nodal properties (2.31) and (2.32) are also each closed conditions in the following sense:

Lemma 2.14 (Closed condition)**.** *Let w_n be a sequence of solutions of* (1.18) *which converges in $C^3_{\mathrm{b}}(\overline{\Omega}) \cap C^1_0(\overline{\Omega})$ to a solution w of* (1.18) *with prescribed pressure R.*

(i) *If the w_n satisfy the nodal elevation properties* (2.31) *and $R_q \leq 0$ for $q \geq 0$, then w also satisfies* (2.31) *unless $w \equiv 0$.*

(ii) *If the w_n satisfy the nodal depression properties* (2.32) *and $R_q \geq 0$ for $q \geq 0$, then w also satisfies* (2.32) *unless $w \equiv 0$.*

Proof. The proofs of (i) and (ii) are the same, so we only prove (i). Thanks to Lemma 2.12, it is enough to show $w_q < 0$ in $\Omega^+ \cup T^+$. By continuity, $w_q \leq 0$ in $\overline{\Omega^+}$. Since $\varphi = w_q$ solves the uniformly elliptic equation (2.33) in Ω^+, $w_q \in C^0_0(\overline{\Omega^+})$, and $w_q \leq 0$ on $B^+ \cup L^+$, the maximum principle Lemma A.14 implies that $w_q < 0$ in $\Omega^+ \cup T^+$ unless $w_q = 0$ at some point $(q^*, 0)$ on T^+. So suppose for contradiction that w_q achieves its maximum value 0 at a point $(q^*, 0)$ on T^+. Differentiating (1.18b) with respect to q we have

$$\frac{w_q w_{qq}}{(H_p + w_p)^2} - \frac{1 + w_q^2}{(H_p + w_p)^3} w_{pq} + g w_q = -R_q \qquad \text{on } T^+. \tag{2.35}$$

Since $w_{qq}(q^*, 0) = 0$, (2.35) gives

$$w_{qp} = R_q (H_p + w_p)^3 \leq 0 \qquad \text{at } (q^*, 0).$$

But the Hopf lemma implies $w_{qp}(q^*, 0) > 0$, a contradiction. $\square$

3 Uniform regularity and decay

In this section we will establish criteria for sequences w_n of solutions to be precompact. In Section 3.1 we will show that $\|w\|_{C^{3+\alpha}}$ is controlled by $\|w_p\|_{L^\infty}$ and $\|R\|_{C^{2+\alpha}}$. While for the periodic problem this would be enough to establish compactness, in an unbounded domain we need additional control at infinity. In Section 3.2 we prove two identities involving a quantity $S(q)$ called the flow force, which we will then use in Section 3.3 to establish uniform decay and precompactness of non-oscillatory sequences of solutions. This is particularly useful since sequences of solutions satisfying either of the nodal properties (2.31) or (2.32) from Section 2.3 cannot oscillate. Most of the results in this section are generalizations to $R \not\equiv 0$ of results in [Whe]. Since the free waves in that paper were all monotone waves of elevation, there was no need to consider oscillatory sequences. As in Section 2, we will work with an arbitrary pressure term R (or a sequence R_n) and not 1-parameter family $R(q;\beta)$.

3.1 Uniform regularity

This section is devoted to the proof of the following proposition:

Proposition 3.1. *For each $K > 0$ there exists a constant C depending only on K such that all solutions w of* (1.18) *with $\|w_p\|_{L^\infty(\Omega)} + \|R\|_{C^{2+\alpha}(\mathbb{R})} \le K$ satisfy $\|w\|_{C^{3+\alpha}(\Omega)} \le C$.*

The proof of Proposition 3.1 is similar to the proof of Proposition 5.12 in [Whe]. The four steps are:

I. Estimate $\|w\|_{C^1}$ in terms of $\|w_p\|_{L^\infty}$ and $\|R\|_{L^\infty}$.
II. Estimate $\|w\|_{C^{2+\alpha'}}$ in terms of $\|w\|_{C^1}$ and $\|R\|_{C^{1+\alpha}}$ for some $\alpha' \in (0, \alpha]$.
III. Estimate $\|w\|_{C^{3+\alpha'}}$ in terms of $\|w\|_{C^{2+\alpha'}}$ and $\|R\|_{C^{2+\alpha'}}$.
IV. Repeat step III with α' replaced by α.

To complete Step I, we use a lower bound on the pressure P. In the stream function formulation, P is given by (1.10),

$$P(x, y) - P_{\mathrm{atm}} = -\frac{|\nabla\psi|^2}{2} - g(y - d) + \Gamma(-\psi) + \frac{\lambda}{2},$$

where (x, y) range over the fluid domain $D_\eta = \{-d < y < \eta(x)\}$. Using (1.12), we can also express P in terms of $w(q, p)$,

$$P(q, p) - P_{\mathrm{atm}} = -\frac{1 + w_q^2}{2(H_p + w_p)^2} - g(H + w - d) + \Gamma + \frac{\lambda}{2}. \tag{3.1}$$

Lemma 3.2. *Suppose that $\eta \in C^2_{\mathrm{b}}(\mathbb{R}) \cap C^0_0(\mathbb{R})$ and $\psi \in C^3_{\mathrm{b}}(\overline{D_\eta})$ satisfy* (1.11)*, $\sup_{D_\eta} \psi_y < 0$, and*

$$\lim_{|x|\to\infty} \sup_y |P(x, y) - P_{\mathrm{atm}} - g(\eta(x) - y)| = 0. \tag{3.2}$$

Then the pressure P is bounded below,

$$P(x, y) - P_{\mathrm{atm}} \ge \inf_{x' \in \mathbb{R}} R(x') - M\psi(x, y), \tag{3.3}$$

where the constant $M = \frac{1}{2}\|\max(\gamma, 0)\|_{L^\infty}$ depends only on γ.

Proof. With $R \equiv 0$, this was proven by Varvaruca [Var09] and also repeated in [Whe]. The modifications for R nonzero are straightforward and hence omitted. □

Using Lemma 3.2 we can now complete Step I:

Proposition 3.3. *Let w be a solution of* (1.18a)–(1.18e). *Then there exist constants $\delta^*, N^* > 0$ depending only on g, γ, d so that*

$$H_p + w_p \geq \frac{\delta^*}{\sqrt{1 + |\inf R|}}, \qquad |w_q| \leq N^* \sqrt{1 + |\inf R|}(1 + |w_p|).$$

In particular, there exists a constant C^* depending only on g, γ, d so that

$$\|w\|_{C^1(\Omega)} \leq C^* \sqrt{1 + |\inf R|}(1 + \|w_p\|_{L^\infty(\Omega)}). \tag{3.4}$$

Proof. Let $\psi \in C^3_{\mathrm{b}}(\overline{D_\eta})$ be the associated stream function. Then $\inf_{D_\eta} \psi_y > 0$ by (1.18d), and (3.2) follows from (3.1) and $w \in C^1_0(\overline{\Omega})$. Letting $M = \frac{1}{2}\|\max(\gamma, 0)\|_{L^\infty}$, Lemma 3.2 therefore implies

$$P - P_{\text{atm}} = -\frac{1 + w_q^2}{2(H_p + w_p)^2} - g(H + w - d) + \frac{1}{2H_p^2} \geq -|\inf R| + Mp,$$

and hence

$$\begin{aligned}\frac{1 + w_q^2}{2(H_p + w_p)^2} &\leq |\inf R| - Mp - g(H + w - d) + \frac{1}{2H_p^2} \\ &\leq |\inf R| - Mp + gd + \frac{1}{2H_p^2} \leq C_1(1 + |\inf R|)\end{aligned} \tag{3.5}$$

where the constant C_1 depends only on g, d, γ. Rearranging (3.5) in two ways, we obtain

$$H_p + w_p \geq \frac{(2C_1)^{-1/2}}{\sqrt{1 + |\inf R|}}, \qquad |w_q| \leq \sqrt{2C_1}\sqrt{1 + |\inf R|}(\|H_p\|_{L^\infty} + |w_p|),$$

as desired. Taking the supremum of both sides of the second inequality yields

$$\|w_q\|_{L^\infty} \leq C\sqrt{1 + |\inf R|}(1 + \|w_p\|_{L^\infty}).$$

Since $w(q,p) = \int_{-m}^0 w_p(q,p)\,dp$ implies $\|w\|_{L^\infty} \leq m\|w_p\|_{L^\infty}$, we therefore have (3.4). $\square$

As in [Whe], we accomplish step II by using a regularity result of Lieberman [Lie87] for fully nonlinear two-dimensional elliptic boundary value problems of the form

$$F(x, D\varphi, D^2\varphi) = 0 \text{ in } B_\rho^-, \qquad G(x, \varphi, D\varphi) = 0 \text{ on } B_\rho^0 := \partial B_\rho^- \cap T. \tag{3.6}$$

Here $B_\rho^- \subset \Omega$ is a half-ball with radius $\rho \in (0, m)$ and center $(q_0, 0) \in T$,

$$B_\rho^- = B_\rho^-(q_0) = \{x \in \mathbb{R}^2 : |x - (q_0, 0)| < \rho,\ p < 0\}, \tag{3.7}$$

and for convenience we have set $x = (q, p)$. We assume that (3.6) is uniformly elliptic with a uniformly oblique boundary condition in the sense that

$$c_1 I \leq F_s(x, r, s) \leq c_2 c_1 I, \qquad |G_r(x, z, r)| \geq c_3 \tag{3.8}$$

for positive constants c_1, c_2, c_3 and all $(x, z, r, s) \in B_\rho^- \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^2$, where $\mathbb{S}^2$ is the space of symmetric 2×2 matrices.

The following is a simplified version of Theorem 3 in [Lie87]:

Theorem 3.4. *Fix $\rho \in (0, m)$ and a Hölder parameter $\alpha \in (0, 1)$, and let $F \in C^{0,1}(\overline{B}_\rho^- \times \mathbb{R}^2 \times \mathbb{S}^2)$ and $G \in C^{1+\alpha}(B_\rho^0 \times \mathbb{R} \times \mathbb{R}^2)$ satisfy* (3.8) *for some positive constants c_1, c_2, c_3. Suppose in addition that there exists a positive constant c_4 so that $\|G\|_{C^{1+\alpha}(B_\rho^0 \times \mathbb{R} \times \mathbb{R}^2)} \leq c_4$ and*

$$|F(x, r, 0)| \leq c_4, \qquad (1 + |s|)|F_r(x, r, s)| + |F_z(x, r, s)| + |F_x(x, r, s)| \leq c_4(1 + |s|^2),$$

for all $(x, r, s) \in B_\rho^- \times \mathbb{R}^2 \times \mathbb{S}^2$*. Then for any* $K > 0$*, there exist positive constants* α' *and* C *depending on* $\alpha, \rho, c_1, c_2, c_3, c_4$ *so that any solution* $\varphi \in C^3(\overline{B}_\rho^-)$ *of* (3.6) *with* $\sup(|\varphi| + |D\varphi|) \le K$ *obeys the estimate*

$$\|\varphi\|_{C^{2+\alpha'}(B^-_{\rho/2})} \le C. \tag{3.9}$$

Using Theorem 3.4, we can now complete step II:

Lemma 3.5. *For each* $K > 0$ *there exists* $C = C(K)$ *and* $\alpha' = \alpha'(K) \in (0, \alpha]$ *so that any solution* w *of* (1.18) *with* $\|w_p\|_{L^\infty(\Omega)} + \|R\|_{C^{1+\alpha}(\mathbb{R})} < K$ *satisfies* $\|w\|_{C^{2+\alpha'}(\Omega)} < C$.

Proof. For convenience set $h = H + w$. In what follows we use $C, \delta > 0$ and $\alpha' \in (0, \alpha]$ to denote constants depending only on K. Proposition 3.3 implies $\|h\|_{C^1(\Omega)} < C$, and Proposition 3.3 implies $\inf_\Omega h_p \ge \delta > 0$.

Now we apply Theorem 3.4. Writing (1.13a)–(1.13b) in non-divergence form, we see that $\varphi = h$ satisfies (3.6) with

$$F(p, r, s) = (1 + r_1^2)s_{22} - 2r_1r_2s_{12} + r_2^2 s_{11} + \gamma(-p)r_2^3$$
$$G(q, z, r) = \frac{1 + r_1^2}{2r_2^2} + g(z - d) - \frac{\lambda}{2} + R(q),$$

for any half-ball $B_1^- = B_1^-(q_0)$, $q_0 \in \mathbb{R}$. Restricting F and G to $|z| + |r| \le K$ and $r_2 \ge \delta$, they satisfy the hypotheses of Theorem 3.4 with c_1, c_2, c_3, c_4 depending on $K, \delta, \alpha, \gamma, \|R\|_{C^{1+\alpha}}$ (but not on q_0). Thus, modifying F and G using cutoff functions and applying Theorem 3.4, we have $\|h\|_{C^{2+\alpha'}(B^-_{1/2})} \le C$ where the constants C and α' only depend on $K, \delta, \|R\|_{C^{1+\alpha}}$.

Letting $G(x, z, p) = p_1$, we can argue similarly for half-balls centered on the bottom boundary $p = -m$. Combining these boundary estimates with $C^{2+\alpha'}$ interior estimates [GT01, Theorem 13.6] for quasilinear equations, we deduce $\|h\|_{C^{2+\alpha'}(\Omega)} < C$ and hence $\|w\|_{C^{2+\alpha'}(\Omega)} < C$ as desired. □

Next we complete step III by differentiating (1.18) with respect to q and applying a Schauder estimate for linear equations:

Lemma 3.6. *For each* $K > 0$ *and* $\alpha' \in (0, \alpha]$ *there exists* $C = C(K, \alpha')$ *so that any solution* w *of* (1.18) *with* $\|w\|_{C^{2+\alpha'}(\Omega)} + \|R\|_{C^{2+\alpha'}(\mathbb{R})} < K$ *satisfies* $\|w\|_{C^{3+\alpha'}(\Omega)} < C$.

Proof. For convenience set $h = H + w$. Differentiating (1.18) with to q, we find that $\varphi = w_q$ satisfies

$$\partial_p\Big(\frac{1 + w_q^2}{h_p^3}\partial_p\varphi - \frac{w_q}{h_p^2}\partial_q\varphi\Big) + \partial_q\Big(-\frac{w_q}{h_p^2}\partial_p\varphi + \frac{1}{h_p}\partial_q\varphi\Big) = 0 \qquad \text{in } \Omega, \tag{3.10}$$

$$\frac{1 + w_q^2}{h_p^3}\partial_p\varphi + \frac{w_q}{h_p}\partial_q\varphi + g\varphi = -R_q \qquad \text{on } p = 0, \tag{3.11}$$

with $\varphi = 0$ on $p = -m$. By Proposition 3.3, $\inf_\Omega h_p \ge \delta$ with $\delta = \delta(K) > 0$. This implies that the operator in (3.10) is uniformly elliptic. Moreover, the boundary condition (3.11) is uniformly oblique in that the coefficient of $\partial_p\varphi$ in (3.11) is uniformly positive. Since the coefficients in (3.10)–(3.11) have their $C^{1+\alpha'}(\Omega)$ norms controlled by K, the Schauder estimate (A.13) therefore gives

$$\|w_q\|_{C^{2+\alpha'}(\Omega)} = C(\|R\|_{C^{1+\alpha'}(\Omega)} + \|w_q\|_{L^\infty(\Omega)}) \le C. \tag{3.12}$$

Solving (1.18a) for w_{pp}, we see that (3.12) implies $\|w_{pp}\|_{C^{1+\alpha'}} < C$, and hence $\|w\|_{C^{3+\alpha'}} < C$. □

Finally, we complete step IV and prove Proposition 3.1.

Proof of Proposition 3.1. Let w solve (1.18) and satisfy $\|w_p\|_{L^\infty} + \|R\|_{C^{2+\alpha}} \le K$. Then Lemma 3.5 implies $\|w\|_{C^{2+\alpha'}} < C$ for some $\alpha' \in (0, \alpha]$, and hence Lemma 3.6 implies $\|w\|_{C^{3+\alpha'}} < C$. Thus $\|w\|_{C^{2+\alpha}} < C$, so another application of Lemma 3.6 gives $\|w\|_{C^{3+\alpha}} < C$. $\square$

3.2 Flow force

In this section we consider the x-dependent quantity

$$S(x) = \int_{-d}^{\eta(x)} \left(P(x,y) - P_{\text{atm}} + (c - u(x,y))^2\right) dy$$

called the *flow force*. In terms of the height function $h = H + w$, Bernoulli constant λ, and antiderivative Γ of the vorticity function, S is given by

$$S(q; h) = \int_{-m}^{0} \left(\frac{1 - h_q^2}{2h_p^2} + \Gamma - g(h - d) + \frac{\lambda}{2}\right) h_p \, dp. \tag{3.13}$$

We view the Bernoulli constant λ in (3.13) as fixed and unrelated to the asymptotic behavior of the height function $h(q,p)$. As in [Whe], we will use the flow force to show that bounded sequences of monotone waves enjoy a uniform decay property.

The following lemma is a consequence of the conservation of x-momentum:

Lemma 3.7. *Let h satisfy* (1.13a)–(1.13c)*. Then*

$$\frac{dS}{dq}(q; h) = R(q) \frac{\partial h}{\partial q}(q, 0). \tag{3.14}$$

In particular, if $R \equiv 0$ then S is constant.

Proof. Let

$$f = \frac{1 - h_q^2}{2h_p^2} + \Gamma - g(h - d) + \frac{\lambda}{2}$$

be one of the factors in the integrand defining S. Then (1.13a) and an integration by parts yield

$$\frac{dS}{dq} = \frac{d}{dq}\int_{-m}^{0} f h_p \, dp = \int_{-m}^{0} (f_q h_p - h_q f_p) \, dp + f h_q\big|_T = -\int_{-m}^{0} \left(\frac{h_q}{h_p^2}\right)_p dp + f h_q\big|_T.$$

By (1.13c), $h_q = 0$ on B, so we conclude

$$\frac{dS}{dq} = \left(-\frac{h_q}{h_p^2} + f h_q\right)\Big|_T = R h_q\big|_T$$

as desired. $\square$

Recall the one-parameter family $\widehat{H}(p; \widehat{\lambda})$ of height functions introduced at the start of Section 2.1,

$$\widehat{H}(p) = \widehat{H}(p; \widehat{\lambda}) = \int_{-m}^{p} \frac{ds}{(\widehat{\lambda} + 2\Gamma(s))^{1/2}}. \tag{3.15}$$

The corresponding flows have constant depth $\widehat{d} = \widehat{H}(0, \widehat{\lambda})$ and total head $\widehat{Q} = \widehat{\lambda}/2 + g\widehat{d}$. From Lemma 2.1 we also have functional relations $\widehat{\lambda} = \widetilde{\lambda}(\widehat{d})$ and $\widehat{Q} = \widetilde{Q}(\widehat{d})$. We define

$$\widetilde{S}(\widehat{d}) = S(q; \widehat{H}(\,\cdot\,; \widetilde{\lambda}(\widehat{d}))), \tag{3.16}$$

where the right hand side is clearly independent of q. Note that in defining $\widetilde{S}(\widehat{d})$ we have *not* replaced the Bernoulli constant λ appearing in (3.13) by $\widetilde{\lambda}(\widehat{d})$.

Lemma 3.8. *The function $\widetilde{S}(\widehat{d})$ defined above satisfies*

$$\widetilde{S}'(\widehat{d}) = \widetilde{Q}(d) - \widetilde{Q}(\widehat{d}). \tag{3.17}$$

In particular, if $0 < \widehat{d} \le d^*$, then $\widetilde{S}(\widehat{d}) > \widetilde{S}(d)$ unless $\widehat{d} = d$. Here d^* was defined in Definition 2.2.

Proof. First we compute

$$\widetilde{S}(\widehat{d}) = \int_{-m}^{0} (\widetilde{\lambda}(\widehat{d}) + 2\Gamma)^{1/2}\,dp + \frac{\lambda - \widetilde{\lambda}(\widehat{d})}{2}\widehat{d} - \frac{g\widehat{d}^2}{2} + gd\widehat{d}.$$

Differentiating with respect to $\widehat{d}$ and using the identity

$$\frac{\mathrm{d}}{\mathrm{d}\widehat{d}} \int_{-m}^{0} (\widetilde{\lambda}(\widehat{d}) + 2\Gamma(p))^{1/2}\,dp = \frac{1}{2}\widehat{d}\,\widetilde{\lambda}'(\widehat{d}),$$

the terms in $\widetilde{S}'(\widehat{d})$ involving derivatives of $\widetilde{\lambda}$ cancel and we are left with (3.17) as desired.

By Lemma 2.1(b), $\widetilde{Q}(\widehat{d})$ is a strictly convex function of $\widehat{d} \in (0, d_{\mathrm{cr}})$, with a unique minimum at $\widehat{d} = d_{\mathrm{cr}} > d$. Recalling the definition of d^*, we deduce

$$\widetilde{Q}(\widehat{d}) < \widetilde{Q}(d) \text{ for } d < \widehat{d} < d^*, \qquad \widetilde{Q}(\widehat{d}) > \widetilde{Q}(d) \text{ for } 0 < \widehat{d} < d.$$

Integrating and applying (3.17), we conclude

$$\widetilde{S}(\widehat{d}) - \widetilde{S}(d) = \int_{d}^{\widehat{d}} (\widetilde{Q}(d) - \widetilde{Q}(s))ds > 0 \quad \text{for } 0 < \widehat{d} \le d^*,\ \widehat{d} \ne d.$$

□

If the vorticity γ is constant, then $\widetilde{S}$ can be computed explicitly,

$$\widetilde{S}(\widehat{d}) = \frac{m^2\widehat{d}}{2d^2} + \frac{m^2}{2\widehat{d}} - \frac{\gamma^2\widehat{d}^3}{24} + \frac{\gamma^2 d^2\widehat{d}}{8} - \frac{g\widehat{d}^2}{2} + gd\widehat{d}. \tag{3.18}$$

In light of Lemma 3.7, one way to obtain (3.18) is to integrate (2.5) and apply (3.14).

We also remark that (3.17) is related to (3.14) in the following way: Suppose that $\widehat{d} = \widehat{d}(q)$ is a slowly varying function. Then $h(q,p) = \widehat{H}(p; \widetilde{\lambda}(\widehat{d}(q)))$ is an approximate solution of (1.13a) and solves the boundary condition (1.13b) exactly with $R(q) = \widetilde{Q}(d) - \widetilde{Q}(\widehat{d}(q))$. If h were an exact solution of (1.13), then (3.14) would give

$$\widetilde{S}'(\widehat{d}(q))\widehat{d}'(q) = S'(q;h) = R(q)\widehat{d}'(q) = (\widetilde{Q}(d) - \widetilde{Q}(\widehat{d}(q)))\widehat{d}'(q),$$

which is (3.17) multiplied by $\widehat{d}'(q)$.

3.3 Uniform decay and compactness

In this section we will show that bounded non-oscillatory sequences of solutions w_n decay uniformly in n as $q \to \pm\infty$. Then we will use Schauder estimates to show that bounded sequences of solutions decaying uniformly at infinity are precompact. First, we restate Definition 1.3 in the (q,p) variables:

Definition 3.9 (Oscillatory sequence)**.** We call sequence of solutions w_n of (1.18) *non-oscillatory* if there exists $M > 0$ such that, for each n, $\partial_q w_n$ does not change sign for $q \ge M$. Otherwise we call w_n *oscillatory.*

Lemma 3.10 (Uniform decay)**.** *Fix an integer $k \ge 2$ and suppose that a sequence (w_n, R_n) of solutions of (1.18) satisfies $\sup_n \|w_n\|_{C^{k+\alpha}(\Omega)} < \infty$ as well as the uniform bounds*

$$\inf_n \inf_q R_n(q) > -\infty, \qquad \int_{\mathbb{R}} \sup_n |R_n(q)|\,dq < \infty, \qquad \lim_{q\to\pm\infty} \sup_n |R_n(q)| = 0, \tag{3.19}$$

on R_n. Then either the w_n satisfy the uniform decay property

$$\lim_{q\to\pm\infty} \sup_n \sup_p \sum_{|\nu|\le k} |D^\nu w_n(q,p)| = 0, \tag{3.20}$$

or w_n is an oscillatory sequence.

Proof. Let w_n be a non-oscillatory sequence, so that there exists $M > 0$ such that each $\partial_q w_n$ does not change sign for $q \ge M$. Assume for contradiction that (3.20) does not hold. Then we can extract a subsequence and find $(q_n, p_n) \in \Omega$ with $q_n \to \infty$ and $p_n \to \bar{p} \in [-m, 0]$ so that

$$\sum_{|\nu|\le k} |D^\nu w_n(q_n,p_n)| \ge \varepsilon \tag{3.21}$$

for all n and some fixed $\varepsilon > 0$. There are two cases: either we can extract a further subsequence so that $\partial_q w_n \ge 0$ for $q \ge M$ for all n, or we can extract a subsequence with $\partial_q w_n \le 0$ for $q \ge M$ for all n. Since the proofs are the same, we will assume we are in the first case.

Set $h_n = H + w_n$, and consider the translated sequence

$$h_n^{(1)}(q,p) = h_n(q + q_n, p).$$

By assumption, $\|h_n\|_{C^{k+\alpha}}$ is bounded uniformly in n, and Proposition 3.3 and the lower bound on R_n in (3.19) together imply $\partial_p h_n \ge \delta$ for all n and some fixed $\delta > 0$. Thus we can extract a subsequence with $h_n^{(1)} \to h^{(1)}$ in $C^k_{\mathrm{loc}}(\overline{\Omega})$, for some $h^{(1)} \in C^{k+\alpha}_{\mathrm{b}}(\overline{\Omega})$ satisfying (1.13a), (1.13c), and $h_p^{(1)} \ge \delta$. Since $\sup_n |R_n(q)| \to 0$ as $q \to \infty$, $h^{(1)}$ also solves (1.13b) with $R = 0$. Lastly, (3.21) guarantees that $h^{(1)}(q,p)$ is not identically H. We will reach a contradiction by showing $h^{(1)} \equiv H$.

Now we compute the flow force $S(q; h^{(1)})$ according to (3.13). From the asymptotic conditions (1.13d) on $h_n^{(1)}$, we know that $S(q; h_n^{(1)}) \to \widetilde{S}(d)$ as $q \to \infty$. Since $h_n^{(1)} \to h^{(1)}$ in $C^k_{\mathrm{loc}}(\overline{\Omega})$, Lemma 3.7 therefore gives

$$S(q;h^{(1)}) - \widetilde{S}(d) = \lim_{n\to\infty} S(q+q_n; h_n^{(1)}) - \widetilde{S}(d) = \lim_{n\to\infty} \int_{q+q_n}^\infty R_n(s)\partial_q h_n(s,0)\,ds$$

for each $q \in \mathbb{R}$. Thus $\sup_n |R_n| \in L^1(\mathbb{R})$ implies

$$|S(q;h^{(1)}(q)) - \widetilde{S}(d)| \le \Big(\sup_n \|\partial_q h_n\|_{L^\infty}\Big) \lim_{n\to\infty} \int_{q+q_n}^\infty \sup_m |R_m(s)|\,ds = 0,$$

i.e. that $S(q; h^{(1)}) \equiv \widetilde{S}(d)$.

Next we use our non-oscillation assumption. Since $\partial_q h_n \ge 0$ for $q \ge M$, we have $h_q^{(1)} \ge 0$ on Ω. Thus

$$h^{(1)}(q,p) \to H_\pm(p) \qquad \text{as } q \to \pm\infty \tag{3.22}$$

pointwise in p for some bounded functions $H_\pm$, and moreover

$$H_-(p) \le h^{(1)}(q,p) \le H_+(p) \qquad \text{in } \overline{\Omega}. \tag{3.23}$$

To get more information about $H_\pm(p)$, we consider the translated sequence

$$h_r^{(2)}(q,p) = h^{(1)}(q + r, p), \qquad r = 1, 2, 3, \dots \tag{3.24}$$

Extracting a subsequence, we can assume that $h_r^{(2)}$ converges in $C^k_{\mathrm{loc}}(\overline{\Omega})$ to a function $h^{(2)} \in C^{k+\alpha}_{\mathrm{b}}(\overline{\Omega})$ solving (1.13a)–(1.13c) with $R = 0$, $h_p^{(2)} \ge \delta$, and $S(q; h^{(2)}) \equiv \widetilde{S}(d)$. But (3.22) implies $h^{(2)} = H_+$, so we deduce that $H_\pm \in C^{k+\alpha}[-m, 0]$ has these properties. Replacing $q + r$ with $q - r$ in (3.24) we obtain the same conclusion for H_-.

Integrating (1.13a) and applying the boundary condition (1.13c), we find $H_\pm(p) = \widehat{H}(p; \widehat{\lambda}_\pm)$ for some Bernoulli constants $\widehat{\lambda}_\pm \geq -2\Gamma_{\min}$. Here the height functions $\widehat{H}$ are defined in (2.1) at the start of Section 2.1. Denoting the corresponding depths by $\widehat{d}_\pm = \widehat{H}(0; \widehat{\lambda}_\pm)$, $S(H_\pm) = \widetilde{S}(d)$ gives

$$\widetilde{S}(\widehat{d}_+) = \widetilde{S}(\widehat{d}_-) = \widetilde{S}(d), \tag{3.25}$$

while the boundary condition (1.13b) gives

$$\widetilde{Q}(\widehat{d}_+) = \widetilde{Q}(\widehat{d}_-) = \widetilde{Q}(d). \tag{3.26}$$

Recalling the definition of d^* (Definition 2.2), we see that (3.26) and the convexity of $\widetilde{Q}$ (Lemma 2.1(b)) imply $\widehat{d}_+, \widehat{d}_- \leq d^*$. Combining this with (3.25), Lemma 3.8 implies $\widehat{d}_+ = \widehat{d}_- = d$ and hence $H_+ = H_- = H$. But then (3.23) forces $h^{(1)} \equiv H$, a contradiction. □

Assuming the uniform decay condition (3.20) in the conclusion of Lemma 3.10, we can obtain compactness using Schauder estimates for linear equations.

Lemma 3.11 (Compactness). *Fix an integer $\ell \geq 2$ and let (w_n, R_n) be a sequence of solutions to* (1.18)*. If $\sup_n \|w_n\|_{C^{\ell+\alpha}(\Omega)} < \infty$, $R_n \to R$ in $C^{\ell-1+\alpha}_{\mathrm{b}}(\mathbb{R})$, and the uniform decay condition* (3.20) *on w_n holds with $k = 0$, then we can extract a subsequence with $w_n \to w$ in $C^{\ell+\alpha}_{\mathrm{b}}(\overline{\Omega})$.*

Proof. As in the proof of Lemma 3.10, Proposition 3.3 guarantees that $\partial_p(H + w_n) \geq \delta > 0$ for some fixed $\delta > 0$. Because of the uniform bounds on $\|w_n\|_{C^{\ell+\alpha}(\Omega)}$ and the uniform decay property (3.20) with $k = 0$, we can then extract a subsequence so that $w_n \to w$ in $C^{\ell}_{\mathrm{loc}}(\overline{\Omega})$ and $C^0_{\mathrm{b}}(\overline{\Omega})$, where $w \in C^{\ell+\alpha}_{\mathrm{b}}(\overline{\Omega})$ and R solve (1.18) and $\partial_p(H + w) \geq \delta$.

Set $h = H + w$ and $h_n = H + w_n$, and let $M = \sup_n \|h_n\|_{C^{\ell+\alpha}(\Omega)} < \infty$. A direct computation shows that $v_n = w_n - w$ satisfies

$$(1 + h_q^2)\partial_p^2 v_n - 2h_p h_q \partial_q \partial_p v_n + 2h_p^2 \partial_q^2 v_n - a_n \partial_q v_n - b_n \partial_p v_n = 0 \qquad \text{in } \Omega, \tag{3.27}$$

$$-c_n \partial_p v_n + d_n \partial_q v_n + g v_n = R - R_n \qquad \text{on } T, \tag{3.28}$$

where the coefficients a_n, b_n are given by

$$\begin{aligned}
a_n &= -h_{nq} h_{npp} - h_q h_{npp} + 2h_{np} h_{npq}, \\
b_n &= -h_{np} h_{nqq} - h_p h_{nqq} + 2h_q h_{npq} - \gamma h_{np}^2 - \gamma h_p h_{np} - \gamma h_p^2, \\
c_n &= \frac{(1 + h_q^2)(h_p + h_{np})}{2h_p^2 h_{np}^2}, \qquad d_n = \frac{h_{nq} + h_q}{2h_{np}^2}.
\end{aligned}$$

Here we've abbreviated $\partial_p h_n$ by h_{np} and so on. Since $h_p \geq \delta$ and $\|h_n\|_{C^{\ell+\alpha}(\Omega)} \leq M$, the elliptic operator in (3.27) is uniformly elliptic and its coefficients are uniformly bounded in $C^{\ell-2+\alpha}_{\mathrm{b}}(\overline{\Omega})$. Likewise the coefficients in (3.28) are uniformly bounded in $C^{\ell-1+\alpha}_{\mathrm{b}}(\overline{\Omega})$, with the coefficient c_n in front of $\partial_p v$ satisfying $c_n \geq \delta M^{-4} > 0$. Thus we have a Schauder estimate (see Lemma A.7)

$$\|v_n\|_{C^{\ell+\alpha}(\Omega)} \leq C(\|R - R_n\|_{C^{\ell-1+\alpha}(\Omega)} + \|v_n\|_{L^\infty(\Omega)}) \xrightarrow{n\to\infty} 0,$$

and hence $w_n \to w$ in $C^{\ell+\alpha}_{\mathrm{b}}(\overline{\Omega})$ as desired. □

4 Continuation

In this section we will prove our global continuation results, Theorems 1.1, 1.2, and 1.4 in the w, q, p variables. See Section 1.3 for the relationship between these and the original u, v, η, P, x, y variables, and [CS04, Whe] for more details about the transformation. As in Section 1.2, we consider a fixed 1-parameter family $\beta \mapsto R(\,\cdot\,;\beta)$ in $C^2(\mathbb{R}, C^{2+\alpha}_{\mathrm{b,e}}(\mathbb{R}))$ of pressure terms satisfying $R(q;0) = 0$ and (1.3):

$$q\beta R_q(q;\beta) \le 0, \qquad \int_{\mathbb{R}} \sup_{|\beta|<M} |R(q;\beta)|\, dq < \infty, \qquad \lim_{\beta\to+\infty} R(0;\beta) = +\infty,$$

for any $M > 0$. The first condition guarantees that sign of R and R_q are both determined by the sign of β. In the simple case where $R(q;\beta) = \beta R^0(q)$, the second condition says that the force $\beta \int_{\mathbb{R}} R^0\, dq$ exerted by the prescribed pressure on a flat free surface is always finite. The last condition guarantees that the limit $\beta \to +\infty$ has physical significance. We note that no assumptions have been made about the behavior of R as $\beta \to -\infty$. One physically meaningful choice would be to have the surface pressure $R(0;\beta)$ at the origin tend to $-\infty$; another would be to have the force $\int_{\mathbb{R}} R(x;\beta)\, dx$ tend to $-\infty$.

In Section 4.1, we will formulate (1.18) as a nonlinear operator equation $\mathscr{F}(\beta, w) = 0$ in a Banach space. We will also state several important results about the corresponding linearized operators. Since R does not appear in the linearized equations, these results were essentially proved already in [Whe], so we will omit almost all of the proofs. In Section 4.2, we will prove Theorem 1.1 using the implicit function theorem together with monotonicity and compactness results from Sections 2 and 3. The remaining theorems are proved using the Healey–Simpson degree [HS98], the important properties of which are summarized in Appendix A.2 for the readers convenience. In Section 4.3, we will define the continua $\mathscr{C}_2^{\pm}$ and $\mathscr{C}_3^{\star}$ in terms of the curve $\mathscr{C}_1$ from Theorem 1.1, making precise the sense in which they are connected. We will then use nodal properties together with a topological degree argument to prove that neither $\mathscr{C}_2^{-}$ nor $\mathscr{C}_3^{\star}$ is precompact. Finally, in Section 4.4 we will prove Theorems 1.2 and 1.4. Unlike in [Whe], we will not use weighted spaces when defining the topological degree for $\mathscr{F}$. This is possible thanks to Lemma 4.12, a simple observation about nonlinear Fredholm operators. We note that Lemma 4.12 can also be applied in [Whe] to remove all mention of weights from the argument.

4.1 Formulation and linearized operators

We define the Banach spaces

$$X = \{w \in C^{3+\alpha}_{\mathrm{b,e}}(\overline{\Omega}) \cap C^2_0(\overline{\Omega}) : w = 0 \text{ on } B\}, \qquad Y = Y_1 \times Y_2,$$
$$Y_1 = C^{1+\alpha}_{\mathrm{b,e}}(\overline{\Omega}) \cap C^0_0(\overline{\Omega}), \qquad Y_2 = C^{2+\alpha}_{\mathrm{b,e}}(T) \cap C^1_0(\overline{\Omega}),$$

where the subscript "e" denotes evenness in q. Because of (1.18d), we introduce a small parameter $\delta \ge 0$ and work in the open subset

$$U_\delta = \left\{ w \in X : \inf_{\Omega}(H_p + w_p) > \delta \right\} \subset X.$$

In this notation, (1.18) is a nonlinear operator equation $\mathscr{F}(\beta, w) = 0$, where

$$\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2)\colon \mathbb{R} \times U_0 \longrightarrow Y$$

is given by

$$\mathscr{F}_1(w) = \Big(-\frac{1+w_q^2}{2(H_p+w_p)^2}+\Gamma\Big)_p + \Big(\frac{w_q}{H_p+w_p}\Big)_q,$$
$$\mathscr{F}_2(\beta,w) = \bigg(\frac{1+w_q^2}{2(H_p+w_p)^2}+gw-\frac{\lambda}{2}+R(\,\cdot\,;\beta)\bigg)\Big|_T.$$

One easily checks that $\mathscr{F}$ is smooth (in fact analytic), and that the linear operator

$$\mathscr{F}_w(w) := \mathscr{F}_w(\beta,w)\colon X \to Y$$

obtained by taking the Fréchet derivative of $\mathscr{F}$ with respect to w is independent of β and given by

$$\mathscr{F}_{1w}(w)\varphi = \Big(\frac{1+w_q^2}{(H_p+w_p)^3}\varphi_p - \frac{w_q}{(H_p+w_p)^2}\varphi_q\Big)_p + \Big(\frac{1}{H_p+w_p}\varphi_q - \frac{w_q}{(H_p+w_p)^2}\varphi_p\Big)_q,$$
$$\mathscr{F}_{2w}(w)\varphi = -\frac{1+w_q^2}{(H_p+w_p)^3}\varphi_p + \frac{w_q}{(H_p+w_p)^2}\varphi_q + g\varphi.$$

In particular, linearizing about the trivial solution $w=0$ we obtain the linear operator from (2.10),

$$\mathscr{F}_{1w}(0)\varphi = \Big(\frac{1}{H_p^3}\varphi_p\Big)_p + \Big(\frac{1}{H_p}\varphi_q\Big)_q, \qquad \mathscr{F}_{2w}(0)\varphi = -\frac{1}{H_p^3}\varphi_p + g\varphi. \tag{4.1}$$

We note that for fixed $w \in U_0$, $\mathscr{F}_{1w}(w)$ is uniformly elliptic and $\mathscr{F}_{2w}(w)$ is uniformly oblique. Indeed, the coefficients of $\mathscr{F}_{1w}(w)$ satisfy

$$\frac{1+w_q^2}{(H_p+w_p)^3}\frac{1}{H_p+w_p} - \bigg(\frac{w_q}{(H_p+w_p)^2}\bigg)^2 = \frac{1}{(H_p+w_p)^4}$$

while the coefficient of φ_p in $\mathscr{F}_{2w}(w)\varphi$ satisfies

$$-\frac{1+w_q^2}{(H_p+w_p)^3} \le -\frac{1}{(H_p+w_p)^3}.$$

Furthermore, $\mathscr{F}_{1w}$ and $\mathscr{F}_{2w}$ are uniformly elliptic and oblique as w ranges over bounded subsets of U_δ with $\delta > 0$ fixed.

We now state several lemmas concerning the linearized operators $\mathscr{F}_w$. These will be needed in Section 4 when we define a topological degree for $\mathscr{F}$. Nearly identical lemmas are proved in [Whe], so we omit almost all of the proofs. While the Hölder spaces $C_0^{k+\alpha}(\overline{\Omega})$ of functions vanishing at infinity do not appear in [Whe], their inclusion poses no difficulty here, see Corollary A.11 and Lemmas A.12 and A.13 in Appendix A.3. As in Appendix A.3, it will sometimes be useful to think of $\mathscr{F}_w$ as a map $X_\mathrm{b} \to Y_\mathrm{b}$, where $X_\mathrm{b}, Y_\mathrm{b}$ are analogues of X, Y without the vanishing condition at infinity or evenness,

$$X_\mathrm{b} = \{w \in C_\mathrm{b}^{3+\alpha}(\overline{\Omega}) : w = 0 \text{ on } B\}, \qquad Y_\mathrm{b} = C_\mathrm{b}^{1+\alpha}(\overline{\Omega}) \times C_\mathrm{b}^{2+\alpha}(T).$$

Lemma 4.1. *Let $w \in U_0$ satisfy*

$$g \sup_q \int_{-m}^0 (H_p+w_p)^3\, dp < 1. \tag{4.2}$$

Then the linear operator $\mathscr{F}_w(w)$ is invertible $X \to Y$ and $X_\mathrm{b} \to Y_\mathrm{b}$. In particular, $\mathscr{F}_w(0)$ is invertible whenever $\lambda > \lambda_\mathrm{cr}$.

Proof. For convenience set $h = H + w$, and consider the Hilbert space

$$\mathscr{H} = \{v \in H^1(\Omega) : u|_B \equiv 0\}.$$

We say that $v \in \mathscr{H}$ is a weak solution of $\mathscr{F}_w(w)v = (f_1, f_2)$ if, for all $\varphi \in \mathscr{H}$,

$$B(v,\varphi) := \iint_\Omega \Big(\frac{1+h_q^2}{h_p^3} v_p\varphi_p - \frac{h_q}{h_p^2}(v_p\varphi_q + v_q\varphi_p) + \frac{1}{h_p} v_q\varphi_q\Big)\,dp\,dq - g\int_T v\varphi\,dx$$

$$= -\iint_\Omega f_1\varphi\,dp\,dq + \int_T f_2\varphi\,dx.$$

Here when integrating by parts we have taken advantage of the special divergence structure of $\mathscr{F}_w(w)$. In particular, we have used that $\mathscr{F}_{2w}(w) - g$ is the conormal derivative operator (see for instance [GT01]).

For smooth $\varphi \in \mathscr{H}$, the elementary bound

$$|\varphi(q,0)|^2 = \left|\int_{-m}^0 \varphi_p(q,p)\,dp\right|^2 \le \left(\int_{-m}^0 h_p^3\,dp\right)\left(\int_{-m}^0 \frac{\varphi_p^2}{h_p^3}\,dp\right)$$

implies

$$\int_T \varphi^2\,dq \le \left(\sup_{q\in\mathbb{R}} \int_{-m}^0 h_p^3\,dp\right) \iint_\Omega \frac{\varphi_p^2}{h_p^3}\,dp\,dq = \rho \iint_\Omega \frac{\varphi_p^2}{h_p^3}\,dp\,dq,$$

where we have defined $\rho = g\sup_q \int_{-m}^0 h_p^3\,dp < 1$. Thus

$$B(\varphi,\varphi) \ge \iint_\Omega \Big(\frac{1-\rho+h_q^2}{h_p^3}\varphi_p^2 - \frac{2h_q}{h_p^2}\varphi_p\varphi_q + \frac{1}{h_p}\varphi_q^2\Big)\,dp\,dq. \tag{4.3}$$

The quadratic form

$$\begin{pmatrix} (1-\rho+h_q^2)/h_p^3 & -h_q/h_p^2 \\ -h_q/h_p^2 & 1/h_p \end{pmatrix}$$

appearing in (4.3) is easily seen to be uniformly positive definite; indeed its diagonal entries as well as its determinant $(1-\rho)/h_p^4$ are uniformly bounded away from zero. Thus (4.3) implies $B(\varphi,\varphi) \ge c\|D\varphi\|^2_{L^2(\Omega)}$. Since functions φ in $\mathscr{H}$ vanish on B, we have $\|\varphi\|_{H^1(\Omega)} \le C\|D\varphi\|_{L^2(\Omega)}$ and hence $B(\varphi,\varphi) \ge c\|\varphi\|^2_{H^1(\Omega)}$. Standard Lax-Milgram arguments then show that $\mathscr{F}_w(w)v = (f_1, f_2)$ has a unique solution $v \in \mathscr{H}$ for any $f \in L^2(\Omega)$ and $g \in L^2(T)$.

See Appendix A in [Whe] for the passage from weak (H^1) to classical ($C_{\mathrm{b}}^{3+\alpha}$) solutions, and Corollary A.11 in this paper for the addition of vanishing conditions at infinity (the limiting operator for $\mathscr{F}_w(w)$ is $\mathscr{F}_w(0)$). To see the second statement in the lemma, we note that, by the definition (1.14) of the Froude number, $w = 0$ satisfies (4.2) if and only if $\lambda > \lambda_{\mathrm{cr}}$. □

As mentioned earlier, the following lemmas appear with only minor modifications in [Whe] and will be given here without proof.

Lemma 4.2 (Index 0)**.** *If $w \in U_0$, then the linear operator $\mathscr{F}_w(w)\colon X \to Y$ is Fredholm with index 0.*

Lemma 4.3. *Fix $w \in U_0$ and set $(A,B) = \mathscr{F}_w(w)$. Then there exists $\kappa_0 < 0$ so that, for $\kappa \in \mathbb{C} \setminus (-\infty, \kappa_0]$, the linear operator $(A - \kappa I, B)\colon X \to Y$ is Fredholm with index 0. Here we temporarily allow functions in X, Y to be complex-valued.*

Lemma 4.4 (Spectral estimate)**.** *Let $\delta > 0$ and let $K \subset U_\delta$ be closed and bounded. Fixing $\theta \in (\pi/2, \pi)$, there exist constants $c_1, c_2 > 0$ such that for all $w \in K$ and $\kappa \in \mathbb{C}$ with $|\arg\kappa| \le \theta$ and $|\kappa| > c_2$,*

$$c_1\|\varphi\|_X \le |\kappa|^{\alpha/2}\|(A-\kappa I)\varphi\|_{Y_1} + |\kappa|^{(1+\alpha)/2}\|B\varphi\|_{Y_2},$$

where $(A,B) = \mathscr{F}_w(w)$.

Definition 4.5. Let $L = (A, B)\colon X \to Y_1 \times Y_2$ be a bounded linear operator between Banach spaces with $X \subset Y_1$. We denote by $\Sigma(A, B)$ the spectrum of A, considered as an unbounded operator $Y_1 \to Y_1$ with domain $X \cap \ker B$.

Lemma 4.6 (Spectral condition)**.** *Let $w \in U_0$ and set $(A, B) = \mathscr{F}_w(w)$. Then there exists an open neighborhood N of the ray $\{\kappa \in \mathbb{C} : \kappa \geq 0\}$ in $\mathbb{C}$ such that $\Sigma(A, B) \cap N$ consists of finitely many eigenvalues, each with finite algebraic multiplicity.*

4.2 First continuation

We now state and prove a more precise version of Theorem 1.1 in terms of the w, q, p variables from Section 1.3. We will abuse notation and continue to refer the curve in Theorem 1.1 as $\mathscr{C}_1$ even when working in the w, q, p variables.

Theorem 4.7. *The trivial solution $(\beta, w) = (0, 0)$ of $\mathscr{F}(\beta, w) = 0$ lies on a unique C^2 curve $\mathscr{C}_1 \subset \mathbb{R} \times U_0$ of solutions parametrized by β,*

$$w = \widetilde{w}(\beta), \qquad \beta_- < \beta < \beta_+, \qquad -\infty \leq \beta_- < 0 < \beta_+ < \infty \tag{4.4}$$

with the following properties. Solutions in $\mathscr{C}_1$ with $\beta > 0$ satisfy the nodal elevation properties (2.31)*, solutions in $\mathscr{C}_1$ with $\beta < 0$ satisfy the nodal depression properties* (2.32)*, and all solutions in $\mathscr{C}_1$ satisfy*

$$g \sup_q \int_{-m}^{0} (H_p + w_p)^3 \, dp < 1. \tag{4.5}$$

Moreover, as $\beta \to \beta_+$ along $\mathscr{C}_1$ either (4.5) *tends to an equality or $\|w_p\|_{L^\infty} \to \infty$. As $\beta \to \beta_-$ we have the same two alternatives or else $\beta_- = -\infty$.*

In the proof, we will need the following elementary observation:

Lemma 4.8. *There exists $\beta^* > 0$ so that any solution (β, w) of $\mathscr{F}(\beta, w) = 0$ has $\beta < \beta^*$.*

Proof. Let (β, w) solve $\mathscr{F}(\beta, w) = 0$, and for convenience set $h = H + w \geq 0$. At $(q, p) = (0, 0)$, the top boundary condition (1.13b) gives

$$-gd \leq \frac{1 + h_q^2}{2h_p^2} + g(h - d) = \frac{\lambda}{2} - R(0; \beta),$$

and hence $R(0; \beta) \leq gd + \frac{\lambda}{2}$. Since $R(0; \beta) \to +\infty$ as $\beta \to \infty$ by (1.3) and g, d, λ are fixed, we can pick $\beta^* > 0$ so that $R(0; \beta) > gd + \frac{\lambda}{2}$ whenever $\beta \geq \beta^*$. $\square$

We also record for later use the following consequence of the results in Section 3:

Proposition 4.9. *Let $(\beta_n, w_n) \in \mathbb{R} \times U_0$ be a sequence of solutions to $\mathscr{F}(\beta, w) = 0$, and suppose that $\sup_n \|\partial_p w_n\|_{L^\infty(\Omega)} < \infty$ and $\inf_n \beta_n > -\infty$. Then either (β_n, w_n) has a convergent subsequence or it is oscillatory according to Definition 3.9.*

Proof. By the elementary Lemma 4.8, $\sup_n \beta_n \leq \beta^* < \infty$, so we can extract a subsequence with $\beta_n \to \beta$ for some $\beta \in \mathbb{R}$. Applying our uniform regularity result Proposition 3.1, we also have $\sup_n \|w_n\|_{C^{3+\alpha}(\Omega)} < \infty$. If (β_n, w_n) is not an oscillatory sequence, then Lemma 3.10 on uniform decay and our assumptions (1.3) on $R(q; \beta)$ imply

$$\lim_{q \to \pm\infty} \sup_n \sup_p |w_n(q, p)| = 0.$$

Applying our compactness result Lemma 3.11, we can therefore extract a further subsequence so that $w_n \to w$ in $C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega})$. $\square$

Proof of Theorem 4.7. By Lemma 4.1, $\mathscr{F}_w(\beta, w) = \mathscr{F}_w(w)$ is invertible for all $w \in U_0$ satisfying (4.5). In particular, $\mathscr{F}_w(\beta, 0)$ is invertible. Therefore by the implicit function theorem there exists a C^2 curve $w = \widetilde{w}(\beta)$ of solutions to $\mathscr{F}(\beta, w) = 0$, defined for $|\beta| < \varepsilon$ for some $\varepsilon > 0$. Unpacking the first statement in (1.3), we find

$$R(q) \geq 0,\ R_q(q) \leq 0 \text{ for } x > 0 \text{ and } \beta > 0, \tag{4.6}$$

$$R(q) \geq 0,\ R_q(q) \leq 0 \text{ for } x > 0 \text{ and } \beta < 0. \tag{4.7}$$

Thus, after possibly shrinking ε, Proposition 2.11 and Lemma 2.12 on nodal properties guarantee that $\widetilde{w}(\beta)$ satisfies the nodal elevation properties (2.31) for $0 < \beta < \varepsilon$ and the nodal depression properties (2.32) for $-\varepsilon < \beta < 0$.

Let (β_-, β_+) be the maximal interval on which a C^2 curve $w = \widetilde{w}(\beta)$ of solutions to $\mathscr{F}(\beta, w)$ satisfying (4.5) can be defined. Our above argument shows that $\beta_- < -\varepsilon < \varepsilon < \beta_+$. We claim that $\widetilde{w}(\beta)$ satisfies the nodal elevation properties (2.31) for $0 < \beta < \beta_+$ and the nodal depression properties (2.32) for $\beta_- < \beta < 0$. Set

$$B_+ = \{\beta \in (0, \beta_+) : \ \widetilde{w}(\beta) \text{ satisfies (2.31)}\},$$

$$B_- = \{\beta \in (\beta_-, 0) : \ \widetilde{w}(\beta) \text{ satisfies (2.32)}\}.$$

We know that each $B_\pm$ is nonempty; indeed we have just shown $B_+ \supset (0, \varepsilon)$ and $B_- \supset (-\varepsilon, 0)$. By Lemma 2.13, $B_\pm$ are each open, and by Lemma 2.14 they are relatively closed. Thus $B_+ = (0, \beta_+)$ and $B_- = (\beta_-, 0)$, which proves the claim.

Now we analyze the limits $\beta \to \beta_\pm$. By Lemma 4.8, β_+ is finite. Suppose for contradiction that

$$\liminf_{\beta \to \beta_+} \|\widetilde{w}_p(\beta)\|_{L^\infty(\Omega)} < \infty, \qquad g \liminf_{\beta \to \beta_+} \sup_q \int_{-m}^0 (H_p + \widetilde{w}_p)^3 \, dp < 1.$$

Then by Proposition 4.9 we can find a sequence $\beta_n \to \beta_+$ with $\widetilde{w}(\beta_n) \to w_+ \in U_0$, where (β_+, w_+) solves $\mathscr{F}(\beta_+, w_+) = 0$ and w_+ satisfies (4.5). By Lemma 4.1, $\mathscr{F}_w(\beta_+, w_+)$ is invertible. But then we can apply the implicit function theorem near (β_+, w_+) to extend our C^2 curve to $\beta > \beta_+$ near β_+ while still maintaining (4.5), contradicting the maximality of β_+. The same argument works as $\beta \to \beta_-$, provided that $\beta_- > -\infty$. ☐

4.3 Nodal properties and compactness

In this section we will prove Proposition 4.11 on the nodal properties of $\mathscr{C}_2^\pm$ and $\mathscr{C}_3^\star$, and also Proposition 4.14, which states that neither $\mathscr{C}_2^-$ nor $\mathscr{C}_3^\star$ is precompact. As in Section 4.2, we will abuse notation and refer to the continua from Theorems 1.2 and 1.4 as $\mathscr{C}_2^\pm$ and $\mathscr{C}_3^\star$ even when working in the w, q, p variables. These continua are defined as follows:

Definition 4.10 (The continua $\mathscr{C}_2^\pm$ and $\mathscr{C}_3^\star$)**.** Let $\mathscr{S}$ be the set of nontrivial solutions of $\mathscr{F}(\beta, w) = 0$,

$$\mathscr{S} = \big\{(\beta, w) \in \mathbb{R} \times U_0 : \mathscr{F}(\beta, w) = 0,\ (\beta, w) \neq (0, 0)\big\},$$

viewed as a subset of $\mathbb{R} \times X$. We define $\mathscr{C}_3^\star$ to be the connected component of $\mathscr{S}$ containing $\mathscr{C}_1^+ = \mathscr{C}_1 \cap \{\beta > 0\}$, and $\mathscr{C}_2^-$ to be the connected component of $\mathscr{S}$ containing $\mathscr{C}_1^- = \mathscr{C}_1 \cap \{\beta < 0\}$. Finally, we define $\mathscr{C}_2^+$ to be the connected component of $\mathscr{C}_3^\star \cap \{\beta \geq 0\}$ containing $\mathscr{C}_1^+$.

Note the inclusions $\mathscr{C}_1^\pm \subset \mathscr{C}_2^\pm$ and $\mathscr{C}_2^+ \subset \mathscr{C}_3^\star$, which are shown in Figure 3a.

Proposition 4.11 (Nodal properties)**.**

(a) *The continua $\mathscr{C}_3^\star$ and $\mathscr{C}_2^-$ are disjoint.*

(b) *All solutions in $\mathscr{C}_2^-$ have $\beta < 0$ and satisfy the nodal depression properties* (2.32)*.*

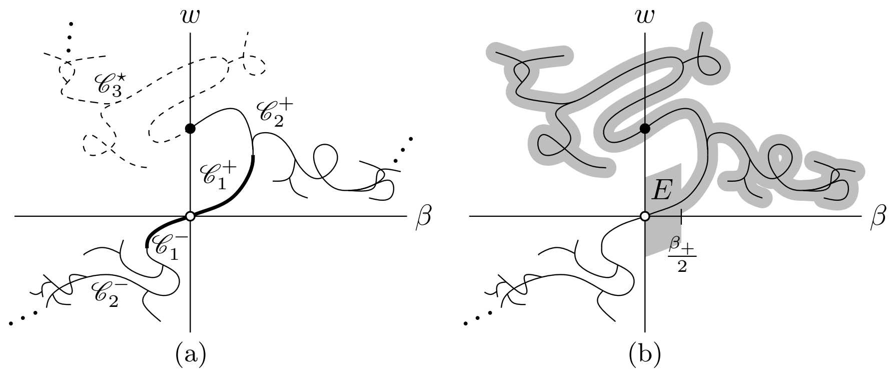

Figure 3: (a) The curves $\mathscr{C}_1^\pm$ and continua $\mathscr{C}_2^\pm$ and $\mathscr{C}_3^\star$ in Definition 4.10. The curves $\mathscr{C}_1^\pm$ are shown in bold, $\mathscr{C}_2^\pm \setminus \mathscr{C}_1^\pm$ with a regular stroke, and $\mathscr{C}_3^\star \setminus \mathscr{C}_2^+$ with dashed lines. (b) The neighborhood V in the proof of Theorem 4.15. V is the shaded region, and E is the strip with $0 < \beta < \frac{1}{2}\beta_+$.

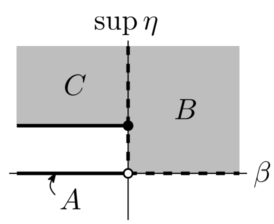

Figure 4: The sets A, B, C from the proof of Proposition 4.11. The dashed lines are not in B.

(c) *All solutions in* $\mathscr{C}_2^+$ *satisfy the nodal elevation properties* (2.31).

(d) *All solutions in* $\mathscr{C}_3^\star \cap \{\beta \le 0\}$ *have* $\sup_T w \ge d^* - d$.

Proof. Recalling that $\eta(q) = w(q, 0)$, consider the continuous function

$$f \colon \mathscr{S} \to \mathbb{R}^2, \qquad f(\beta, w) = (\beta, \sup_T w).$$

Combining Corollaries 2.5 and 2.6, we find that $f(\mathscr{S}) \subset A \cup B \cup C$, where

$$A = \{(\beta, 0) : \beta < 0\}, \quad B = \{(\beta, s) : \beta, s > 0\}, \quad C = \{(\beta, s) : \beta \le 0,\ s \ge d^* - d\}.$$

These sets are depicted in Figure 4. Here A corresponds to waves of depression with $\beta < 0$, B corresponds to waves of elevation with $\beta > 0$, and C corresponds to waves with $\beta \le 0$ which are not waves of depression. From Theorem 4.7, we know that $\mathscr{C}_1^-$ meets A and $\mathscr{C}_1^+$ meets B. Therefore $\mathscr{C}_2^-$ meets A and $\mathscr{C}_3^\star$ meets B. Since A and $B \cup C$ form a separation of $A \cup B \cup C$ and $\mathscr{C}_2^-$ and $\mathscr{C}_3^\star$ are connected, we deduce that $\mathscr{C}_2^- \subset f^{-1}(A)$ and $\mathscr{C}_3^\star \subset f^{-1}(B \cup C)$. In particular, $\mathscr{C}_2^-$ and $\mathscr{C}_3^\star$ are disjoint.

Next, note that $f(\mathscr{C}_2^-) \subset A$ implies $\mathscr{C}_2^- \subset \{\beta < 0\}$. We already know that solutions in $\mathscr{C}_1^- \subset \mathscr{C}_2^-$ satisfy the nodal depression properties (2.32). Since $\mathscr{C}_2^-$ is connected, Lemmas 2.13 and 2.14 on the preservation of nodal properties then imply that all solutions in $\mathscr{C}_2^-$ satisfy (2.32). When applying these lemmas it is essential that β not change sign on $\mathscr{C}_2^-$. Similarly, since $\mathscr{C}_2^+ \subset \{\beta \ge 0\}$ is connected and contains the curve $\mathscr{C}_1^+$ along which the nodal elevation properties (2.31) hold, (2.31) holds for all solutions in $\mathscr{C}_2^+$. Finally, we observe that $\mathscr{C}_3^\star \cap \{\beta \le 0\} \subset f^{-1}(C)$, and hence that solutions in $\mathscr{C}_3^\star \cap \{\beta \le 0\}$ have $\sup_T w \ge d^* - d$. □

In order to prove Proposition 4.14, we will use the Healey–Simpson degree summarized in Section A.2. To define this degree for our nonlinear operator $\mathscr{F}(\beta, w)$, we need to show that a suitable restriction of $\mathscr{F}$ is proper. We call a nonlinear mapping $F\colon X \to Y$ *proper* if $F^{-1}(A)$ is compact whenever $A \subset Y$ is compact. We will show that $\mathscr{F}$ is proper using the following simple lemma.

Lemma 4.12. *Let X, Y be Banach spaces, $U \subset X$ an open set, and $F\colon U \to Y$ a C^1 mapping whose Fréchet derivative $F_x(x)$ is Fredholm for each $x \in U$. Let $K \subset U \cap F^{-1}(0)$, and denote by K_ε the ε-neighborhood of K. If K is compact, then $F|_{\overline{K_\varepsilon}}\colon \overline{K_\varepsilon} \to Y$ is proper for all $\varepsilon > 0$ sufficiently small.*

Proof. First fix $x^* \in K$. Since $L = F_x(x^*)$ is Fredholm, we can decompose $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ where $X_2 = \ker L$ and $Y_1 = \operatorname{ran} L$. Using these decompositions we can rewrite the equation $F(x) = y$ as the system

$$F_1(x_1, x_2) = y_1, \tag{4.8}$$

$$F_2(x_1, x_2) = y_2, \tag{4.9}$$

where $D_{x_1} F_1(x_1^*, x_2^*) = L|_{X_1}\colon X_1 \to Y_1$ is invertible. The implicit function theorem then allows us to solve (4.8) for x_1, that is there exists a neighborhood $V^* \times W^*$ of $(x^*, 0) \in U \times Y$ together with a C^1 function $\widetilde{x}_1(x_2, y_1)$ so that

$$\big\{(x,y) \in \overline{V^* \times W^*} : F_1(x_1, x_2) = y_1\big\} = \big\{(x,y) \in \overline{V^* \times W^*} : x_1 = \widetilde{x}_1(x_2, y_1)\big\}. \tag{4.10}$$

We claim that F is proper when restricted to $\overline{V^*} \cap F^{-1}(W^*)$. Suppose that a sequence $x^n \in \overline{V^*}$ has $F(x^n) = y^n \to y$ with $y^n \in \overline{W^*}$. We need to show that x^n has a convergent subsequence. Since F is Fredholm, X_2 is finite-dimensional, so we can extract a subsequence with $x_2^n \to x_2$ in X_2. But then (4.10) gives

$$x_1^n = \widetilde{x}_1(x_2^n, y_1^n) \to \widetilde{x}_1(x_2, y_1)$$

and hence that $x^n = (x_1^n, x_2^n)$ is convergent, proving the claim. Since $\overline{V^*} \cap F^{-1}(\overline{W^*})$ contains the open neighborhood $V^* \cap F^{-1}(W^*)$ of x^*, F is therefore proper when restricted to $\overline{B_\varepsilon}(x^*)$ for $\varepsilon > 0$ sufficiently small.

Since K is compact, we can therefore find finitely many points $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $K \subset \cup_i B_\varepsilon(x_i)$ and F is proper when restricted to each $\overline{B_{2\varepsilon}}(x_i)$. Therefore F is proper when restricted to the union $\cup_i \overline{B_{2\varepsilon}}(x_i)$. Lastly, we note that $F^{-1}(0) \subset \cup_i B_\varepsilon(x_i)$ implies

$$K_\varepsilon = \{x \in U : \operatorname{dist}(x, K) < \varepsilon\} \subset \cup_i B_{2\varepsilon}(x_i),$$

and hence that F is proper when restricted to $\overline{K_\varepsilon}$. □

Lemma 4.13. *Let $K \subset (\mathbb{R} \times U_0) \cap \mathscr{F}^{-1}(0)$ be compact, and let K_ε denote the ε-neighborhood of K. Then for all $\varepsilon > 0$ sufficiently small, $\mathscr{F}|_{\overline{K_\varepsilon}}$ is an admissible generalized homotopy (Definition A.5) with parameter β.*

Proof. First we claim that for $(\beta, w) \in \mathbb{R} \times U_0$, the linear operator $(A, B) = \mathscr{F}_w(\beta, w)$ is admissible according to Definition A.2. Condition (i) is Lemma 4.2, condition (iii) is a special case of Lemma 4.4, and condition (iv) is Lemma 4.6. Finally, condition (ii) is a consequence of condition (iv): By (iv), there exists $\kappa \in \mathbb{C}$ such that $(A - \kappa I, B)$ is onto $X \to Y_1 \times Y_2$. Thus $B\colon X \to Y_2$ must be onto.

Next we claim that, for ε sufficiently small, $\mathscr{F}(\beta, \cdot)\colon \overline{K_\varepsilon} \to Y$ is admissible according to Definition A.3. We have just shown (ii). By the compactness of $K \subset \mathbb{R} \times U_0$, we can pick ε small enough that $\overline{K_\varepsilon} \subset \mathbb{R} \times U_\delta$ for some $\delta > 0$. In particular $\mathscr{F}|_{\overline{K_\varepsilon}}$ is C^2, which implies (i). By the compactness of K and Lemma 4.12, we can then shrink ε further so that $\mathscr{F}|_{\overline{K_\varepsilon}}$ is proper, which implies (iii). Comparing with Definition A.5, we conclude that $\mathscr{F}|_{\overline{K_\varepsilon}}$ is an admissible generalized homotopy. □

We are now ready to prove Proposition 4.14, which will be the core of our proofs of Theorems 1.2 and 1.4. It is also the only place where we use the topological degree.

Proposition 4.14. *Neither $\mathscr{C}_3^\star$ nor $\mathscr{C}_2^-$ is precompact.*

Proof. As in [Whe, Theorem 5.2], we follow the proof of Theorem II.6.1 in [Kie04]. Assume for contradiction that $\mathscr{C}_3^\star$ is precompact. Then

$$K := \overline{\mathscr{C}_3^\star} = \mathscr{C}_3^\star \cup \{(0,0)\}$$

is compact. Letting K_ε denote the ε-neighborhood of K, Lemma 4.13 guarantees that there exists $\varepsilon_0 > 0$ such that $\mathscr{F}|_{\overline{K_{\varepsilon_0}}} \colon \overline{K_{\varepsilon_0}} \to Y$ is an admissible generalized homotopy with parameter β.

Let $\beta_\pm$ and $\widetilde{w}(\beta)$ be as in Theorem 4.7. Then, for all $\beta_- < \beta < \beta_+$, $\mathscr{F}(\beta, \widetilde{w}(\beta)) = 0$ and $\mathscr{F}_w(\beta, \widetilde{w}(\beta))$ is invertible. Thus by the implicit function theorem we can find $0 < \varepsilon_1 < \varepsilon_0/2$ so that $w = \widetilde{w}(\beta)$ whenever $\mathscr{F}(\beta, w) = 0$, $\|w - \widetilde{w}(\beta)\|_X < 2\varepsilon_1$, and $\frac{1}{2}\beta_- \le \beta \le \frac{1}{2}\beta_+$. Define the strip

$$E = \left\{(\beta, w) : 0 < \beta < \tfrac{1}{2}\beta_+,\ \|w - \widetilde{w}(\beta)\|_X < \varepsilon_1\right\}.$$

Using the notation $A_\beta = \{w : (\beta, w) \in A\}$ for subsets A of $\mathbb{R} \times X$, the above properties of $\mathscr{F}$ imply

$$\deg(\mathscr{F}(\beta, \,\cdot\,), E_\beta, 0) = (-1)^{\nu(\beta)} \ne 0, \qquad 0 < \beta < \tfrac{1}{2}\beta_+, \tag{4.11}$$

where "deg" is the Healey–Simpson degree summarized in Section A.2 and $\nu(\beta)$ is the number of positive eigenvalues of $\mathscr{F}_w(\beta, \widetilde{w}(\beta))$ counted according to algebraic multiplicity. We will reach a contradiction by using additivity and homotopy invariance (Lemmas A.4 and A.6) to show that the degree in (4.11) vanishes.

We claim that $\mathscr{C}_3^\star \setminus E \subset K$ is closed and hence compact. Since $\mathscr{C}_3^\star$ is a connected component of $\mathscr{F}^{-1}(0) \setminus \{(0,0)\}$, it is enough to show that $(0,0)$ is not a limit point of $\mathscr{C}_3^\star \setminus E$. So let $(\beta_n, w_n) \in \mathscr{C}_3^\star$ be a sequence with $\beta_n \ne 0$ which converges to $(0,0)$. Applying the implicit function theorem near $(0,0)$, we can assume without loss of generality that $(\beta_n, w_n) \in \mathscr{C}_1$. Since $\mathscr{C}_1 \cap \{\beta < 0\}$ is contained in $\mathscr{C}_2^-$, which is disjoint from $\mathscr{C}_3^\star$ by Proposition 4.11(a), we must have $\beta_n > 0$. But then $(\beta_n, w_n) \in E$ for n sufficiently large, which proves the claim.

Since the compact set $\mathscr{C}_3^\star \setminus E$ does not meet either of the closed sets $\mathscr{F}^{-1}(0) \setminus \mathscr{C}_3^\star$ or

$$\partial_w E = \left\{(\beta, w) : 0 < \beta < \widetilde{\beta}_+,\ \|w - \widetilde{w}(\beta)\|_X = \varepsilon_1\right\},$$

we can find $0 < \varepsilon_2 < \varepsilon_1$ such that $\operatorname{dist}(\mathscr{C}_3^\star \setminus E, \partial_w E)$ and $\operatorname{dist}(\mathscr{C}_3^\star \setminus E, \mathscr{F}^{-1}(0) \setminus \mathscr{C}_3^\star)$ are both greater than ε_2. Letting V^1 be the ε_2-neighborhood of $\mathscr{C}_3^\star \setminus E$, we define the bounded open subsets

$$V := V^1 \cup E, \qquad W := V^1 \setminus \overline{E}$$

of K_{ε_1}. See Figure 3b for an illustration of E and V. We claim that V and W satisfy

$$E \subset V \text{ and } \overline{V} \subset K_{\varepsilon_0}, \tag{4.12a}$$

$$V \cap \partial_w E = \varnothing, \tag{4.12b}$$

$$\mathscr{F} \ne 0 \text{ on } \partial V \setminus \{(0,0)\}, \tag{4.12c}$$

$$\mathscr{F} \ne 0 \text{ on } \partial W \setminus \{(\tfrac{1}{2}\beta_+, \widetilde{w}(\tfrac{1}{2}\beta_+))\}, \tag{4.12d}$$

$$V_\beta = W_\beta \cup E_\beta \text{ and } W_\beta \cap E_\beta = \varnothing \text{ for } 0 < \beta < \tfrac{1}{2}\beta_+. \tag{4.12e}$$

The first three properties (4.12a)–(4.12c) are straightforward, and (4.12e) is a consequence of (4.12b). To see the remaining property (4.12d), observe that $(0,0) \in \mathscr{F}^{-1}(0) \setminus \mathscr{C}_3^\star$ is a positive distance away from V^1.

We now compute the degree of $\mathscr{F}$ on sections of V and W. By (4.12a) and (4.12c), homotopy invariance (Lemma A.6) implies that $\deg(\mathscr{F}(\beta, \,\cdot\,), V_\beta, 0)$ is constant for $\beta > 0$. Since $V_\beta = \varnothing$ for β sufficiently large and positive, this gives

$$\deg(\mathscr{F}(\beta, \,\cdot\,), V_\beta, 0) = 0 \qquad \text{for } \beta > 0.$$

Similarly by (4.12a) and (4.12d), $\deg(\mathscr{F}(\beta,\,\cdot\,),W_\beta,0)$ is constant for $\beta<\frac{1}{2}\beta_+$. Since $W_\beta=\varnothing$ for β sufficiently large and negative, this yields

$$\deg(\mathscr{F}(\beta,\,\cdot\,),W_\beta,0)=0 \qquad \text{for } \beta<\tfrac{1}{2}\beta_+.$$

Finally, by additivity (Lemma A.4) and (4.12e),

$$\begin{aligned}\deg(\mathscr{F}(\beta,\,\cdot\,),E_\beta,0)&=\deg(\mathscr{F}(\beta,\,\cdot\,),V_\beta,0)-\deg(\mathscr{F}(\beta,\,\cdot\,),W_\beta,0) \qquad \text{for } 0<\beta<\tfrac{1}{2}\beta_+\\ &=0-0=0,\end{aligned}$$

contradicting (4.11).

The argument for $\mathscr{C}_2^-$ is similar. □

4.4 Second and third continuation

In this final section we will state and prove more precise versions of Theorems 1.2 and 1.4 in the w,q,p variables. The proofs are straightforward, relying on Proposition 4.9 on compactness, Proposition 4.11 on nodal properties, and Proposition 4.14 on noncompactness. See Definition 4.10 for the definitions of the continua $\mathscr{C}_2^\pm$ and $\mathscr{C}_3^\star$.

Theorem 4.15. *Solutions in $\mathscr{C}_2^-$ have $\beta<0$ and satisfy the nodal depression properties* (2.32)*. Moreover, $\mathscr{C}_2^-$ satisfies one of the two alternatives*

(i$^-$) *(Stagnation)* $\sup_{\mathscr{C}_2^-}\|w_p\|_{L^\infty(\Omega)}=\infty$*; or*

(ii$^-$) *(β large and negative)* $\inf_{\mathscr{C}_2^-}\beta=-\infty$.

Solutions in $\mathscr{C}_2^+$ satisfy the nodal elevation properties (2.31)*, and $\mathscr{C}_2^+$ satisfies one of the two alternatives*

(i$^+$) *(Stagnation)* $\sup_{\mathscr{C}_2^+}\|w_p\|_{L^\infty(\Omega)}=\infty$*; or*

(ii$^+$) *(Free wave) There exists a solution other than $(\beta,w)=(0,0)$ in $\mathscr{C}_2^+$ with $\beta=0$.*

Proof. The statements about nodal properties as well as the containment $\mathscr{C}_2^-\subset\{\beta<0\}$ are already proved in Proposition 4.11, so it only remains show the alternatives. Suppose first that neither of the alternatives (i$^-$), (ii$^-$) hold for $\mathscr{C}_2^-$. Since the nodal depression properties (2.32) hold along $\mathscr{C}_2^-$, it cannot contain any oscillatory sequences (Definition 3.9). But then Proposition 4.9 implies that $\mathscr{C}_2^-$ is precompact, contradicting Proposition 4.14.

Now suppose that neither of the alternatives (i$^+$), (ii$^+$) hold for $\mathscr{C}_2^+$. Then $\mathscr{C}_2^+\subset\{\beta>0\}$. Since $\mathscr{C}_2^+$ is the connected component of $\mathscr{C}_3^\star\cap\{\beta\ge 0\}$ containing $\mathscr{C}_1^+$, this implies that $\mathscr{C}_3^\star=\mathscr{C}_2^+\subset\{\beta>0\}$. In particular, $\mathscr{C}_3^\star$ satisfies the nodal elevation properties (2.31) by Proposition 4.11. Applying Proposition 4.9 as before, we deduce that $\mathscr{C}_2^+=\mathscr{C}_3^\star$ is precompact, violating Proposition 4.14. □

Theorem 4.16. *Consider the same situation as in Theorem 4.15. Suppose that alternative* (i$^+$) *does not hold, so that $\mathscr{C}_2^+$ meets $\{\beta=0\}$. Then*

(a) *Some solutions in $\mathscr{C}_3^\star$ have $\beta<0$, and all solutions in $\mathscr{C}_3^\star\cap\{\beta<0\}$ have $\sup_T w\ge d^*-d$. In particular, $\mathscr{C}_3^\star$ does not meet $\mathscr{C}_2^-$.*

(b) *One of the following three alternatives holds:*

(iii) *(Stagnation)* $\sup_{\mathscr{C}_3^\star\setminus\mathscr{C}_2^+}\|w_p\|_{L^\infty(\Omega)}=\infty$*; or*

(iv) *(β large and negative)* $\inf_{\mathscr{C}_3^\star \setminus \mathscr{C}_2^+} \beta = -\infty$*; or*

(v) *(Oscillation)* $\mathscr{C}_3^\star \setminus \mathscr{C}_2^+$ *contains an oscillatory sequence* (β_n, w_n)*.*

(c) $\mathscr{C}_2^+$ *is precompact.*

Proof. By Theorem 4.15, we know that solutions in $\mathscr{C}_2^+$ satisfy the nodal elevation properties (2.31). Since (i$^+$) does not hold and $\mathscr{C}_2^+ \subset \{\beta \geq 0\}$, Proposition 4.9 then implies that $\mathscr{C}_2^+$ is precompact, which is (c).

By Proposition 4.11, will have (a) if we can show that $\mathscr{C}_3^\star \cap \{\beta < 0\}$ is nonempty. If it were empty, we would have $\mathscr{C}_3^\star \subset \{\beta \geq 0\}$ and hence $\mathscr{C}_3^\star = \mathscr{C}_2^+$. But then $\mathscr{C}_3^\star$ would be precompact by (c), contradicting Proposition 4.14. To prove (b), assume for contradiction that none of the alternatives (iii), (iv), (v) hold. Then $\mathscr{C}_3^\star \setminus \mathscr{C}_2^+$ is precompact by Proposition 4.9. But then (c) implies that $\mathscr{C}_3^\star = \mathscr{C}_2^+ \cup (\mathscr{C}_3^\star \setminus \mathscr{C}_2^+)$ is also precompact, again contradicting Proposition 4.14. □

A Appendix

A.1 Examples

In this appendix we will construct an explicit family of solutions of (1.11) with constant vorticity γ and for certain surface pressures $R(x;\beta)$. Solutions in this family with $\beta > 0$ are monotone waves of elevation, and those with $\beta < 0$ are monotone waves of depression. There are solutions with β arbitrarily large and negative, as well as solutions with $\beta > 0$ which are arbitrarily close to stagnation, either at their crests or at the point on the bed below the crest. In this sense both alternatives (ii$^-$) and (i$^+$) from Theorem 1.2 occur.

In Lemma A.1 below, we define these solutions and give several important formulas and qualitative properties. In Section A.1.1, we will compare Lemma A.1 with Section 1.2 and Theorem 1.2. Finally, we will prove Lemma A.1 in Section A.1.2. Recall the definition $D_\eta = \{(x, y) : -d < y < \eta(x)\}$ of the fluid domain from Section 1.2.

Lemma A.1. *Consider the stream function*

$$\psi(x, y) = \psi(x, y; \beta) = \frac{\beta}{2} \log \frac{(y + d - b)^2 + x^2}{(y + d + b)^2 + x^2} - (\mu + d\gamma)y - \frac{\gamma y^2}{2}, \tag{A.1}$$

where $\beta \in \mathbb{R}$ is a parameter and γ, d, b, μ are constants satisfying $0 < d < b$, $\mu > 0$, and $\mu + \gamma d > 0$. Also fix the gravitational constant $g > 0$, and set $\beta_B = b\mu/2$ so that $\psi_y(0, -d; \beta_B) = 0$. Then

(a) *There exists $\beta_T > 0$ and a smooth, strictly increasing function*

$$\eta_0 \colon (-\infty, \beta_T) \to (-d, -d + b)$$

satisfying $\eta_0(0) = 0$ and

$$\psi(0, \eta_0(\beta); \beta) = 0, \qquad \lim_{\beta \to -\infty} \eta_0(\beta) = -d, \qquad \lim_{\beta \to -\infty} \psi_y(0, \eta_0(\beta); \beta) = -\infty,$$
$$\psi_y(0, \eta_0(\beta); \beta) < 0, \qquad \lim_{\beta \to \beta_T} \eta_0(\beta) \in (0, -d + b), \qquad \lim_{\beta \to \beta_T} \psi_y(0, \eta_0(\beta); \beta) = 0.$$

(b) *For all $-\infty < \beta < \min(\beta_T, \beta_B)$ there exists an even free surface profile $\eta(\,\cdot\,;\beta) \in C_0^\infty(\mathbb{R})$ with $\eta(0;\beta) = \eta_0(\beta)$ so that $\psi(\,\cdot\,;\beta) \in C_{\mathrm{b}}^\infty(\overline{D_\eta})$ solves (1.11) with surface pressure $R(\,\cdot\,;\beta) \in C_0^\infty(\mathbb{R})$ defined by (1.11d). Here the asymptotic shear flow U, flux m, Bernoulli constant λ, and Froude number F are given by*

$$c - U(y) = \gamma(y + d) + \mu, \quad m = \frac{\gamma d^2}{2} + \mu d, \quad \lambda = (\gamma d + \mu)^2, \quad F = \frac{\sqrt{d\gamma\mu + \mu^2}}{\sqrt{dg}}. \tag{A.2}$$

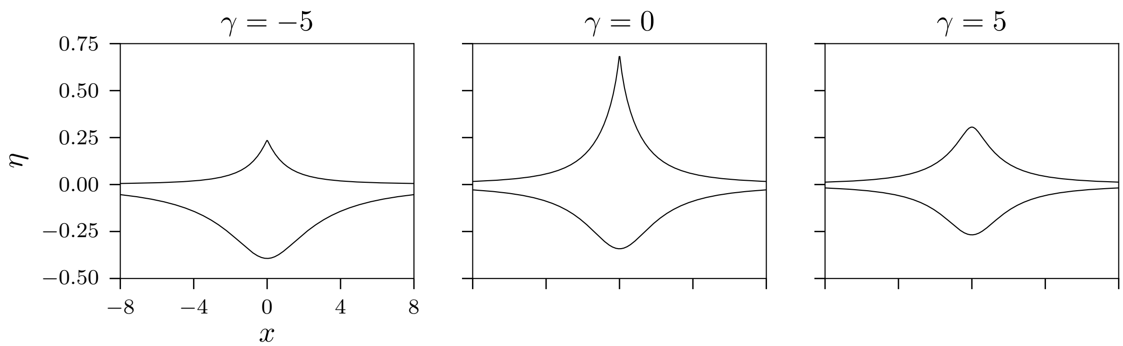

Figure 5: In each plot, the upper curve is the free surface $y = \eta(x;\beta)$ with $\beta = \min(\beta_T, \beta_B)$, and the lower curve is the free surface with $\beta = -1.5$. Here $d = 1$, $g = 1$, $b = 2$, $F = 3$, and $\mu \approx 2.57$. Other streamlines of the taller waves are shown in Figure 6.

Moreover, η and R are C^∞_{b} functions of $(x,\beta) \in \mathbb{R} \times I$ for any interval I compactly contained in $(-\infty, \min(\beta_T, \beta_B))$.

(c) *The solutions from* (b) *satisfy the no-stagnation condition $\sup_{D_\eta} \psi_y < 0$ as well as the nodal properties*

$$\beta\eta_{xx}(0;\beta) < 0,\ \ \beta x \eta_x < 0,\ \ \beta x \psi_x < 0 \qquad \text{for } -d < y \le \eta(x;\beta) \text{ and } \beta x \neq 0. \tag{A.3}$$

If $\beta = 0$ then $\eta, R, \psi_x \equiv 0$.

(d) *The value $R(0;\beta)$ of the surface pressure at $x = 0$ tends to $-\infty$ as $\beta \to -\infty$.*

A.1.1 Comparison with Section 1.2

Consider the same situation as in Lemma A.1, and let

$$(u, v, \eta) = (u(\beta), v(\beta), \eta(\beta)), \qquad -\infty < \beta < \min(\beta_T, \beta_B),$$

be the smooth curve of solutions of (1.1) given by $\psi_x = -v$ and $\psi_y = u - c$. Defining the connected sets

$$\begin{aligned} \mathscr{C}_2^- &= \{(\beta, u(\beta), v(\beta), \eta(\beta)) : -\infty < \beta < 0\}, \\ \mathscr{C}_2^+ &= \{(\beta, u(\beta), v(\beta), \eta(\beta)) : 0 < \beta < \min(\beta_B, \beta_T)\}, \end{aligned}$$

we have by Lemma A.1(c) that $\mathscr{C}_2^-$ consists of monotone waves of depression with $\beta < 0$ while $\mathscr{C}_2^+$ consists of monotone waves of elevation with $\beta > 0$. We plot some typical free surfaces in Figure 5. Since there is a solution $(\beta, u(\beta), v(\beta), \eta(\beta))$ in $\mathscr{C}_2^-$ for all $\beta < 0$, $\mathscr{C}_2^-$ satisfies alternative (ii$^-$) of Theorem 1.2.

We claim that alternative (i$^+$) holds for $\mathscr{C}_2^+$. Note that $\sup_{D_\eta} u < c$ for each fixed $\beta < \min(\beta_B, \beta_T)$ by Lemma A.1(c). If $\beta_B < \beta_T$, then as $\beta \to \beta_B$ the wave $(u, v, \eta)(\beta)$ approaches stagnation at the point $(-d, 0)$ on the bed directly below the crest,

$$\sup_{D_\eta}(u - c) \ge u(0, -d) - c = \psi_y(0, -d) = \frac{2\beta}{b} - \mu \nearrow 0.$$

On the other hand if $\beta_T \ge \beta_B$, then as $\beta \to \beta_T$ the wave $(u, v, \eta)(\beta)$ approaches stagnation at its crest,

$$\sup_{D_\eta}(u - c) \ge u(\eta(0), -d) - c = \psi_y(\eta_0(\beta), -d; \beta) \to 0$$

by Lemma A.1(a). Streamlines for some limiting waves with stagnation points are shown in Figure 6.

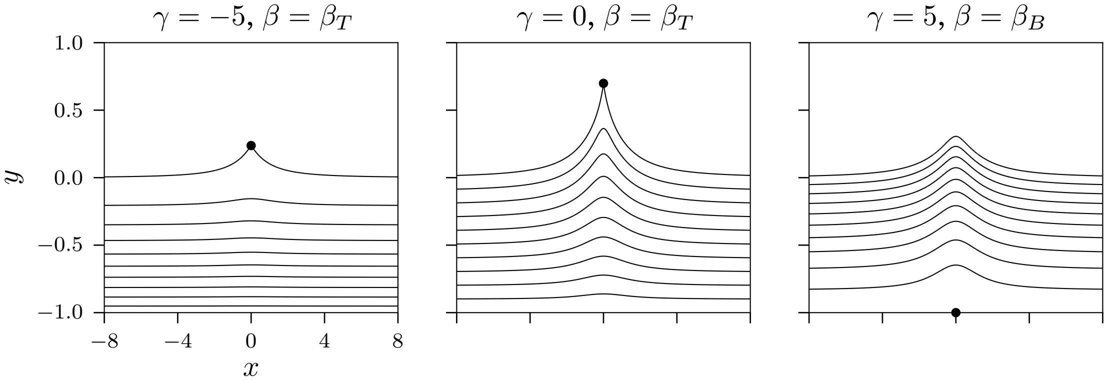

Figure 6: Streamlines for the tallest waves in Figure 5. Stagnation points are indicated by dots.

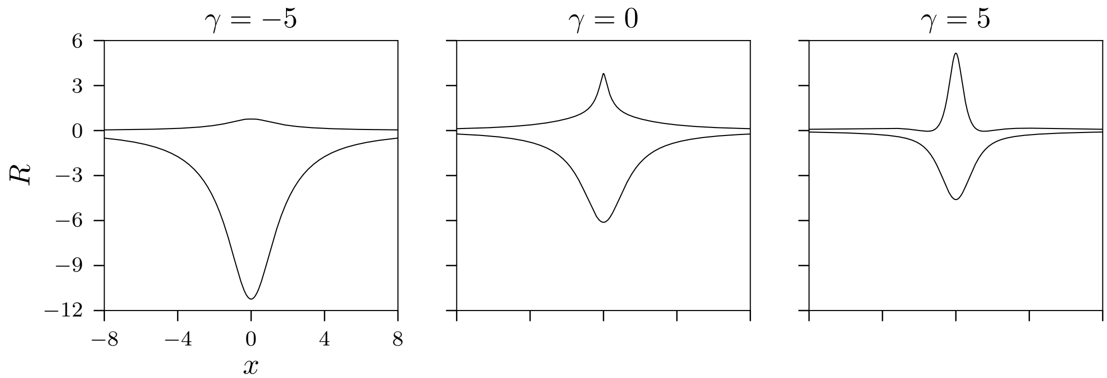

Figure 7: The surface pressures $R = R(x;\beta)$ for the waves whose free surfaces η are plotted in Figure 5. The upper curves have $\beta = \min(\beta_T, \beta_M)$ while the lower curves have $\beta = -1.5$.

Next we consider the hypotheses set out in Section 1.2 on the shear flow U and Froude number F. From (A.2) we know that $c - U(y) = \gamma(y + d) + \mu$ is linear, so U certainly has the regularity $U \in C^{3+\alpha}[-d, 0]$ for any α. Moreover, the maximum value of $U(y) - c$ on $[-d, 0]$ is $\max(-\mu, -\mu - \gamma d)$, which is negative. However, the assumptions in Lemma A.1 do not imply that the Froude number is supercritical. Using (A.2) we see that $F > 1$ is equivalent to $\mu(\mu + d\gamma) > 1$. (The influence of the Froude number on the surface pressure R can be seen in (A.6) below.)

The remaining hypotheses in Section 1.2 concern the surface pressures $R(x;\beta)$. We plot some typical surface pressures in Figure 7. We have already shown in Lemma A.1 that R is smooth jointly in x and β and that $R(x;0) \equiv 0$. Since $R(\,\cdot\,;\beta)$ is only defined for $\beta < \min(\beta_T, \beta_B)$, we cannot talk about the limit of $R(0;\beta)$ as $\beta \to +\infty$; the limit of $R(0;\beta)$ as $\beta \to \min(\beta_T, \beta_B)$ is easily seen to be finite. The last two hypotheses on R are

$$x\beta R_x(x;\beta) \le 0, \tag{A.4}$$

$$\int_{\mathbb{R}} \sup_{|\beta|<M} |R(x;\beta)|\,dx < \infty \quad \forall M > 0. \tag{A.5}$$

Taylor expanding (A.1) and (1.11d) near $y = 0$ and $x = \pm\infty$, we find

$$\eta(x;\beta) = \frac{2bd\beta}{d\gamma + \mu} \cdot \frac{1}{x^2} + O\Big(\frac{1}{x^4}\Big), \qquad R(x;\beta) = g(F^2 - 1)\eta(x;\beta) + O\Big(\frac{1}{x^4}\Big), \tag{A.6}$$

as $x \to \pm\infty$, and similarly

$$\eta_x(x;\beta) = \frac{2bd\beta}{d\gamma+\mu}\cdot\frac{1}{x^2} + O\Big(\frac{1}{x^5}\Big), \qquad R_x(x;\beta) = g(F^2-1)\eta_x(x;\beta) + O\Big(\frac{1}{x^5}\Big), \tag{A.7}$$

where the implied constants are uniform for β in any compact subset I of $(-\infty, \min(\beta_B,\beta_T))$. From the first equation (A.6) we have

$$\int_{\mathbb{R}} \sup_{\beta\in I}|R(x;\beta)|\,dx < \infty,$$

which is similar to (A.5). Also, if $F>1$, then by (A.7) there exists $M = M(I) > 0$ so that (A.4) holds for $\beta \in I$ and $|x| > M$.

Finally, we numerically investigate the condition (A.4) on $R(x;\beta)$ for finite values of x. Plotting $R(x;\beta)$ for several supercritical Froude numbers $F>1$ and a variety of values of d, b, μ, γ, g, it seems that (A.4) always holds when $\beta < 0$. On the other hand, (A.4) does not always seem to hold when β is sufficiently close to $\min(\beta_T,\beta_B)$. This is easy to see in the third plot in Figure 7: the upper curve $R(x;\beta_B)$ has two local minima at $x \approx \pm 1.55$. Next we consider the upper curves in the other two plots. These surface pressures $R(x;\beta_T)$ correspond to the flows in Figure 6 with stagnation points at their crests. Zooming in very closely near the stagnation point, we appear to have $\eta(x) \approx \eta(0) - k|x|$ and $R(x) \approx R(0) + gk|x|$ for some positive constant k. While not visible in Figure 7, this again seems to violate (A.4). The constant $k = |\psi_{xx}/\psi_{yy}|^{1/2}(0,\eta(0))$ is predicted by a Taylor expansion.

A.1.2 Proof of Lemma A.1

In this section we prove Lemma A.1. The core of the proof is showing that $\eta_0(\beta)$ and $\eta(x;\beta)$ exist and have the desired properties. Though we will not make use of this fact, we note that the inverse functions of $x \mapsto \eta(x;\beta)$ and $\beta \mapsto \eta_0(\beta)$ both have explicit formulas.

Proof of Lemma A.1. We begin by proving (a). Computing

$$\psi(0,0;0) = 0, \qquad \psi_y(0,0;0) = -\mu - d\gamma < 0, \qquad \psi_\beta(0,y;\beta) > 0 \text{ for } -d < y < -d+b,$$

we have by the implicit function theorem there exists a smooth increasing function $\eta_0(\beta)$ solving $\psi(0,\eta_0(\beta);\beta) = 0$, defined for $|\beta|$ sufficiently small and with $\eta_0(0) = 0$. By a standard argument based on the implicit function theorem, we can then extend the domain of definition of η_0 to some (possibly infinite) interval (β_-,β_T) such that η_0 is smooth and strictly increasing on (β_-,β_T), and such that $\beta_- < 0 < \beta_T$ are maximal in the following sense: As $\beta \to \beta_-$ we have either $\psi_y(0,\eta_0(\beta);\beta) \to 0$ or $\eta_0(\beta) \to -d$ or else $\beta_- = -\infty$, while as $\beta \to \beta_T$ we have either $\psi_y(0,\eta_0(\beta);\beta) \to 0$ or $\eta_0(\beta) \to -d+b$ or else $\beta_- = +\infty$.

First consider the limit $\beta \to \beta_-$. Now $\eta_0(\beta)$ is strictly decreasing, $\eta_0(\beta) < 0$ for all $\beta_- < \beta < 0$. For any $\beta < 0$ and $-d < y < 0$ we have

$$\psi_y(0,y;\beta) \le -\gamma(y+d) - \mu \le -|\gamma|d - \mu < 0,$$

so this means that $\psi_y(0,\eta_0(\beta);\beta)$ is uniformly bounded above for $\beta_- < \beta < 0$, which rules out $\psi_y(0,\eta_0(\beta);\beta) \to 0$ as $\beta \to \beta_-$. Since $\psi(0,-d;\beta) = m > 0$ for all β, the continuity of ψ also rules out $\eta_0(\beta) \to -d$ as $\beta \to \beta_-$ unless $\beta_- = -\infty$. Therefore $\beta_- = -\infty$. Taking limits in (A.1), we see that $\beta \to -\infty$ forces $\eta_0(\beta) \to -d$. Taylor expanding, we find

$$\eta_0(\beta) = -d - \frac{bm}{2\beta} + O\Big(\frac{1}{\beta^2}\Big), \qquad \psi_y(0,\eta_0(\beta);\beta) = \frac{2\beta}{b} - \mu + O\Big(\frac{1}{|\beta|}\Big)$$

as $\beta \to -\infty$, and hence $\psi_y(0,\eta(\beta);\beta) \to -\infty$. Next we consider the limit $\beta \to \beta_T$. For β large enough, $\inf_{-d<y<-d+b}\psi(0,y;\beta) > 0$, so β_T must be finite. Since $\psi(0,y;\beta_T) \to +\infty$ as

$y \to -d+b$, we also cannot have $\eta_0(\beta) \to -d+b$ as $\beta \to \beta_T$. The only remaining possibility is that $\psi_y(0;\eta_0(\beta);\beta) \to 0$ as $\beta \to \beta_T$.

Fix $-\infty < \beta < \min(\beta_B, \beta_T)$. We claim that $\sup_S \psi_y < 0$, where S is the strip

$$S := \{(x,y) : -d < y < \eta_0^+\}, \qquad \text{where } \eta_0^+ = \max(\eta_0, 0),\ \eta_0^- = \min(\eta_0, 0).$$

Note that S is nonempty and that $\eta_0 < -d+b$ guarantees $\psi \in C_{\mathrm{b}}^\infty(\overline{S})$. Factoring the rational expressions for the relevant partials, we find

$$\beta x \psi_x \le 0 \text{ and } \beta x \psi_{xy} \le 0 \text{ for } -d < y < -d+b, \text{ with equality iff } x\beta = 0. \tag{A.8}$$

First consider the case where $\beta > 0$. Then $\eta_0 > 0$ means $\eta_0^+ = \eta_0$, and $\beta x \psi_{xy} \le 0$ implies that $\sup_{\overline{S}} \psi_y$ is achieved along the vertical line $x = 0$. Checking that $\psi_y(0,y)$ is a convex function of $-d < y < \eta_0$, we deduce

$$\sup_S \psi_y = \max\{\psi_y(0,-d), \psi_y(0,\eta_0)\}. \tag{A.9}$$

Since $\beta < \beta_B$, we have $\psi_y(0,-d) = 2\beta/b - \mu < 0$, and by part (a) we have $\psi_y(0,\eta_0) < 0$. Thus (A.9) implies $\sup_S \psi_y < 0$. Now consider the case where $\beta \le 0$. Then $\eta_0^+ = 0$, and $\beta x \psi_{xy} \le 0$ implies that $\sup_{\overline{S}} \psi_y$ is achieved as $x \to \pm\infty$. Since $\lim_{x\to\pm\infty} \psi_y(0,y)$ is linear, we have

$$\sup_S \psi_y = \max\left\{ \lim_{x\to\pm\infty} \psi_y(0,-d), \lim_{x\to\pm\infty} \psi_y(0,\eta_0) \right\} = \max\{-\mu, -\mu - \gamma d\} < 0,$$

thanks to our initial assumptions on μ, γ, d.

Still with β fixed, we next prove the existence of $\eta(x)$ solving $\psi(x,\eta(x)) = 0$ and satisfying the nodal properties in (c). If $\beta = 0$ then $\psi(x,0) \equiv 0$ and we can take $\eta \equiv \eta_0 = 0$, so assume $\beta \neq 0$. Since $\psi(0,\eta_0) = 0$ and $\psi_y(0,\eta_0) < 0$, we can find a smooth function $\eta(x)$ solving $\psi(x,y) = 0$ for $|x|$ sufficiently small, with $\eta(0) = \eta_0$. Since $\psi_x = 0$ for $x = 0$, we have $\eta_x(0) = 0$. To extend this local solution, we need the following additional facts about ψ:

$$\beta\psi(x,0) > 0, \qquad \lim_{x\to\pm\infty} \psi(x,0) = 0. \qquad \beta\psi_{xx}(0,y) < 0 \text{ for } -d < y \le \eta_0, \tag{A.10}$$

which can be checked by direct computation. By the last inequality in (A.10), $\beta\eta_{xx}(0) < 0$, so $\eta_0^- < \eta(x) < \eta_0^+$ for $|x| \neq 0$ sufficiently small. Thus by the first inequality in (A.8), $\beta x \eta_x(x) < 0$ for $|x| \neq 0$ sufficiently small. So suppose that this solution $\eta(x)$ can be smoothly extended to an interval $(-L, L)$, with

$$\eta_0^- < \eta(x) < \eta_0^+ \text{ and } \beta x \eta_x(x) < 0 \text{ for } 0 < |x| < L, \tag{A.11}$$

and that $(-L, L)$ is the maximal interval for which this is possible. We claim that $L = +\infty$. Assume not. Since η is strictly increasing or decreasing as $x \to \pm L$, it has a limit

$$y^* = \lim_{x\to\pm L} \eta(x) \in [\eta_0^-, \eta_0^+],$$

and by continuity $\psi(L, y^*) = 0$. Since $\inf_S \psi_y < 0$ implies $\psi_y(\pm L, y^*) < 0$, we can therefore apply the implicit function theorem near $(\pm L, y^*)$ to slightly extend the domain of definition of η. By the first inequality in (A.8), $\beta\eta_x(L) < 0$, so this extension can be made to preserve the second inequality in (A.11). Thus by the maximality of L, the first set of inequalities in (A.11) must be violated, i.e. we must have $y^* \in \{\eta_0^-, \eta_0^+\}$. If $\beta < 0$, then η is increasing as a function of $|x|$, so the only possibility is $y^* = \eta_0^+ = 0$. Similarly, if $\beta > 0$ then η is a decreasing function of $|x|$, so the only possibility is $y^* = \eta_0^- = 0$. But by the first inequality in (A.10), $\psi(L, 0) \neq 0$, so in either case we have contradicted $\psi(L, y^*) = 0$. Thus the claim is proved: $\eta(x)$ is defined and smooth for all $x \in \mathbb{R}$ and satisfies the inequalities in (A.11) for $x \neq 0$. Moreover, η satisfies all of the nodal properties (A.3).

With $\eta(x;\beta)$ defined, the rest of the lemma is straightforward. Let $-\infty < \beta < \min(\beta_T, \beta_B)$. First we claim that $\eta \to 0$ as $x \to \pm\infty$. By (A.3), $\eta(x;\beta)$ has a limit $y^*(\beta) \in [\eta_0^-(\beta), \eta_0^+(\beta)]$ as $x \to \pm\infty$. The continuity and decay properties of ψ imply $\psi(x, y^*;\beta) \to 0$ as $x \to \infty$, which then forces $y^*(\beta) = 0$, proving the claim. Therefore $\overline{D_\eta} \subset \overline{S}$ and $\psi \in C_{\mathrm{b}}^\infty(\overline{D_\eta})$. Thus $\sup_S \psi_y < 0$ implies $\sup_{D_\eta} \psi_y < 0$. We also compute $\Delta\psi = -\gamma$ on $\overline{D_\eta}$, which is (1.11a). The formulas in (A.2) for the flux m, Bernoulli constant λ, and Froude number F in (A.2) are easily verified using their definitions in (1.2) and (1.6) together with the formula for $c - U$, and the bottom boundary condition (1.11b) is immediate. The limits $\psi_x \to 0$ and $\psi_y \to U - c$ as $x \to \pm\infty$ are uniform for $-d \le y \le \eta_0^+$, so the asymptotic conditions (1.11e) are satisfied.

Differentiating $\psi(x, \eta(x;\beta);\beta) = 0$ with respect to x, we deduce $\eta \in C_0^\infty(\mathbb{R})$. The surface pressure R is defined by the boundary condition (1.11d),

$$R(x;\beta) = -\tfrac{1}{2}|\nabla\psi|^2(x, \eta(x;\beta);\beta) - g\eta(x;\beta) + \tfrac{1}{2}\lambda.$$

Since η, ψ_x, and $\psi_y^2 - \lambda$ are all $C_0^\infty(\mathbb{R})$ for each fixed β, we have $R(\,\cdot\,;\beta) \in C_0^\infty(\mathbb{R})$ as well. Also, (a) implies

$$R(0;\beta) = -\tfrac{1}{2}\psi_y^2(0, \eta(0;\beta);\beta) - g\eta(0;\beta) + \tfrac{1}{2}\lambda \to -\infty \qquad \text{as } \beta \to -\infty,$$

which is (d). The smooth dependence of R and η on β from part (b) is clear from their construction. □

A.2 The Healey–Simpson degree

In this appendix we repeat a summary [CS04, Whe] of the key features of the Healey–Simpson degree [HS98] for the reader's convenience. This degree is needed for the proof of Proposition 4.14. First we define a notion of admissibility for linear maps, taken from [HS98, CS04].

Definition A.2. Let X, Y_1, Y_2 be Banach spaces, with X continuously embedded in Y_1, and set $Y = Y_1 \times Y_2$. We call a bounded linear operator $L = (A, B)\colon X \to Y$ *admissible* if

(i) L is a Fredholm operator of index zero.

(ii) B is surjective.

(iii) There exist constants $c_1, c_2 > 0$ and $\alpha \in (0,1)$ such that

$$c_1\|u\|_X \le |\kappa|^{\alpha/2}\|(A - \kappa I)u\|_{Y_1} + |\kappa|^{(1+\alpha)/2}\|Bu\|_{Y_2}$$

for all $u \in X$ and real $\kappa \ge C_2$.

(iv) There exists an open neighborhood N of the ray $\{\mu : \mu \ge 0\} \subset \mathbb{C}$ such that $\Sigma(A,B) \cap N$ consists of finitely many eigenvalues, each of finite algebraic multiplicity. Here, as in Definition 4.5, $\Sigma(A,B)$ denotes the spectrum of A considered as an unbounded operator $\tilde{A}\colon X \to Y$ with domain $D(\tilde{A}) = X \cap \ker B$.

We next define admissibility for nonlinear operators, again following [HS98, CS04]. We call a nonlinear mapping $F\colon X \to Y$ *proper* if $F^{-1}(K)$ is compact whenever $K \subset Y$ is compact. We call F *locally proper* if $D \cap F^{-1}(K)$ is compact whenever $K \subset Y$ is compact and $D \subset X$ is closed and bounded. Note that all proper maps are locally proper.

Definition A.3. In the setting of Definition A.2, let $W \subset X$ be open and bounded. A map $F = (F_1, F_2)\colon \overline{W} \to Y$ is *admissible* if

(i) $F \in C^2(W, Y) \cap C^0(\overline{W}, Y)$.

(ii) For each $u \in \overline{W}$, $F_u(u)$ is admissible according to Definition A.2.

(iii) $F\colon \overline{W} \to Y$ is locally proper.

Suppose that $F\colon \overline{W} \to Y$ is admissible and $y \in Y \setminus F(\partial W)$ is a regular value of F, so that $F_u(u)$ is invertible for all $u \in F^{-1}(y) \cap W$. Then $F^{-1}(y) \cap W$ is finite, and we define

$$\deg(F, W, y) = \sum_{u \in F^{-1}(y) \cap W} (-1)^{\nu(u)},$$

where $\nu(u)$ is the number, counted according to algebraic multiplicity, of positive eigenvalues in $\Sigma(F_u(u))$, which is finite by admissibility, and where the sum over the empty set is 0. If $y \notin F(\partial W)$ is not a regular value, we define $\deg(F, W, y)$ to be $\deg(F, W, \widetilde{y})$ for some nearby regular value $\widetilde{y}$ which exists by the Sard–Smale theorem; see [HS98].

We need two properties of the degree. The first is additivity:

Lemma A.4 (Additivity)**.** *Suppose that $W, V \subset X$ are bounded open sets with $W \cap V = \varnothing$ and that $F\colon \overline{W} \cup \overline{V} \to Y$ is admissible. If $y \notin F(\partial W \cup \partial V)$, then*

$$\deg(F, W \cup V, y) = \deg(F, W, y) + \deg(F, V, y).$$

The second property is homotopy-invariance, proven in [HS98]:

Definition A.5 (Admissible generalized homotopy)**.** For $\Upsilon \subset [0,1] \times W$ open, we say that $H\colon \overline{\Upsilon} \to Y$ is an *admissible generalized homotopy* if $H \in C^2(\Upsilon, Y)$ is locally proper and $H(t, \cdot)$ is admissible for each t. We call $t \in [0,1]$ the parameter of the homotopy.

For $\Upsilon \subset [0,1] \times W$ and $t \in [0,1]$, set $\Upsilon_t = \{u \in W : (t,u) \in \Upsilon\}$.

Lemma A.6 (Homotopy invariance)**.** *If $H\colon \overline{\Upsilon} \to Y$ is an admissible generalized homotopy, and $y \notin H(\partial \Upsilon_t)$ for $t \in [0,1]$, then* $\deg(H(0, \cdot), \Upsilon_0, y) = \deg(H(1, \cdot), \Upsilon_1, y)$.

A.3 Elliptic problems in infinite strips

In this appendix we give several slight variations of standard facts about elliptic problems in infinite strips, mostly without proofs.

Consider a linear equation

$$Au = a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } \Omega, \quad Bu = \sigma^i D_i u + \mu u = g \text{ on } T, \quad u = 0 \text{ on } B, \tag{A.12}$$

in an infinite strip $\Omega = \{(q,p) \in \mathbb{R}^2 : -m < p < 0\}$ with width $m > 0$, top boundary $T = \{p = 0\}$, and bottom boundary $B = \{p = -m\}$. Here $D_1 = \partial_q$, $D_2 = \partial_p$, and we are using the usual summation convention. We assume that A is uniformly elliptic and B is uniformly oblique in that $a^{ij} = a^{ji}$, $a^{ij}\xi_i\xi_j \geq c|\xi|^2$, and $|\sigma^2| \geq c$ for some constant $c > 0$. We also assume that the coefficients have the regularity $a^{ij}, b^i, c \in C^{k-2+\alpha}_{\mathrm{b}}(\overline{\Omega})$ and $\mu, \sigma^i \in C^{k-1+\alpha}_{\mathrm{b}}(T)$ for some integer $k \geq 2$ and $\alpha \in (0,1)$.

Such equations enjoy the following classical Schauder estimate:

Lemma A.7 (Schauder estimate, cf. [ADN59])**.** *If $u \in C^{k+\alpha}_{\mathrm{b}}(\overline{\Omega})$ of solves* (A.12)*, then*

$$\|u\|_{C^{k+\alpha}(\Omega)} \leq C(\|f\|_{C^{k-2+\alpha}(\Omega)} + \|g\|_{C^{k-1+\alpha}(T)} + \|u\|_{L^\infty(\Omega)}), \tag{A.13}$$

where the constant C depends only on the ellipticity and obliqueness constants and the stated norms of the coefficients.

Defining the Banach spaces

$$X_{\mathrm{b}} = \{u \in C^{k+\alpha}_{\mathrm{b}}(\overline{\Omega}) : u|_B \equiv 0\}, \qquad Y_{\mathrm{b}} = C^{k-2+\alpha}_{\mathrm{b}}(\overline{\Omega}) \times C^{k-1+\alpha}_{\mathrm{b}}(T), \tag{A.14}$$

the pair (A, B) defines a bounded linear operator $L = (A, B)\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$. Recall that L is called *locally proper* if $D \cap L^{-1}(K)$ is compact whenever $K \subset Y_{\mathrm{b}}$ is compact and $D \subset X$ is closed and bounded. Since L is linear, it is locally proper if and only if it has closed range and finite dimensional kernel, i.e. if it is semi-Fredholm with index $< +\infty$. While the Schauder estimate Lemma A.7 is sufficient to prove local properness for linear elliptic operators in bounded domains, in an unbounded domain we need additional conditions at infinity, see [VV03].

We now assume that the coefficients in (A.12) have limits as $q \to \pm\infty$,

$$a^{ij}(q,p) \to \widetilde{a}^{ij}(p), \quad b^i(q,p) \to \widetilde{b}^i(p), \quad c(q,p) \to \widetilde{c}(p), \quad \mu(q) \to \widetilde{\mu}, \quad \sigma^i(q) \to \widetilde{\sigma}^i,$$

and that the limiting coefficients have the regularity $\widetilde{a}^{ij}, \widetilde{b}^i, \widetilde{c} \in C^{k-2+\alpha}[-m, 0]$. We then define the limiting operator

$$\widetilde{L} = (\widetilde{A}, \widetilde{B}), \qquad \widetilde{A}u = \widetilde{a}^{ij} D_{ij} u + \widetilde{b}^i D_i u + \widetilde{c} u, \qquad \widetilde{B}u = \widetilde{\sigma}^i D_i u + \widetilde{\mu} u.$$

We gave a proof of the following lemma in [Whe], which in turn was a simplified version of the much more general proof of Volpert and Volpert in [VV03].

Lemma A.8. *Assume the homogeneous limiting problem $\widetilde{L}u = 0$ has no nontrivial solutions $u \not\equiv 0$ in X_{b}. Then $L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is locally proper.*

Lemma A.8 is easily extended to closed subspaces $X_0 \subset X_{\mathrm{b}}$, $Y_0 \subset Y_{\mathrm{b}}$ of functions vanishing at infinity: Fix an integer $2 \le \ell \le k$, and define

$$\begin{aligned} X_0 &= X_{\mathrm{b}} \cap C_0^{\ell}(\overline{\Omega}), & Y_{0,1} &= C_{\mathrm{b}}^{k-2+\alpha}(\overline{\Omega}) \cap C_0^{\ell-2}(\overline{\Omega}), \\ Y_0 &= Y_{0,1} \times Y_{0,2}, & Y_{0,2} &= C_{\mathrm{b}}^{k-1+\alpha}(T) \cap C_0^{\ell-1}(T). \end{aligned}$$

One easily checks that $L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ restricts to a bounded linear operator $X_0 \to Y_0$. Moreover, since X_0, Y_0 are closed subspaces, we can immediately prove the following:

Lemma A.9. *Suppose that $L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is locally proper. Then $L\colon X_0 \to Y_0$ is also locally proper.*

Proof. Let $x_n \in X_0$ be a bounded sequence with $Lx_n = y_n \to y$ in Y_0. Since $L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is locally proper, we can extract a subsequence so that $x_n \to x \in X_{\mathrm{b}}$. Since X_0 is a closed subspace, we have $x \in X_0$. □

Next we turn to invertibility and Fredholm properties of $L\colon X_0 \to Y_0$. The first step is the following lemma, which is proved using a translation argument (of the sort used to prove Lemma A.8).

Lemma A.10. *Assume the homogeneous limiting problem $\widetilde{L}u = 0$ has no nontrivial solutions $u \not\equiv 0$ in X_{b}. Then $L^{-1}(Y_0) \subset X_0$, i.e. if $Lu = (f, g) \in Y_0$ for some $u \in X_{\mathrm{b}}$, then in fact $u \in X_0$.*

Proof. Suppose that $Lu = (Au, Bu) = (f, g) \in Y_0$ for some $u \in X_{\mathrm{b}}$. If $u \notin X_0$, then there exists a sequence $(q_n, p_n) \in \overline{\Omega}$ with $|q_n| \to \infty$ such that

$$\sum_{r=0}^{\ell} |D^r u(q_n, p_n)| \ge \delta$$

for some fixed $\delta > 0$. Consider the shifted functions

$$u_n(q,p) = u(q + q_n, p), \qquad f_n(q,p) = f(q + q_n, p), \qquad g_n(q) = g(q + q_n),$$

and the shifted operators

$$\begin{aligned} L_n &= (A_n, B_n), \\ A_n \varphi &= a^{ij}(q + q_n, p) D_{ij}\varphi + b^i(q + q_n, p) D_i \varphi + c(q + q_n, p)\varphi, \\ B_n \varphi &= \sigma^i(q + q_n) D_i \varphi + \mu(q + q_n)\varphi. \end{aligned}$$

Extracting a subsequence, we can assume that $p \to p^* \in [0,d]$. Since $\|u_n\|_{C^{k+\alpha}(\Omega)} = \|u\|_{C^{k+\alpha}(\Omega)} < \infty$, we can also extract a further subsequence so that $u_n \to v$ in $C^k_{\mathrm{loc}}(\overline{\Omega})$, where $v \in C^{k+\alpha}_{\mathrm{b}}(\overline{\Omega})$. Since $f \in C^{\ell-2+\alpha}_0(\overline{\Omega})$ and $g \in C^{\ell-1+\alpha}_0(T)$, we can assume that $f_n \to 0$ in $C^{\ell-2}_{\mathrm{loc}}(\overline{\Omega})$ and $g_n \to 0$ in $C^{\ell-1}_{\mathrm{loc}}(T)$. Finally, we can extract yet again so that $L_n u_n \to \widetilde{L} v$ in $C^{\ell-2}_{\mathrm{loc}}(\overline{\Omega}) \times C^{\ell-1}_{\mathrm{loc}}(T)$. Since $L_n u_n = (f_n, g_n) \to 0$ in $C^{\ell-2}_{\mathrm{loc}}(\overline{\Omega}) \times C^{\ell-1}_{\mathrm{loc}}(T)$, this means $\widetilde{L} v = 0$ and hence $v \equiv 0$. But

$$\sum_{r=0}^{\ell} |D^r v(0,p^*)| = \lim_{n\to\infty} \sum_{r=0}^{\ell} |D^r u(q_n,p^*)| \geq \delta,$$

so $v \not\equiv 0$, a contradiction. □

Invertibility of $L\colon X_0 \to Y_0$ is then an easy corollary:

Corollary A.11. *Assume the homogeneous limiting problem $\widetilde{L}u = 0$ has no nontrivial solutions $u \not\equiv 0$ in X_{b} and that $L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is invertible. Then $L\colon X_0 \to Y_0$ is also invertible.*

Proof. Since $X_0 \subset X_{\mathrm{b}}$, $L\colon X_0 \to Y_0$ is clearly one-to-one. To see that it is onto, let $y \in Y_0$. Since $L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is invertible, there exists $x \in X_{\mathrm{b}}$ with $Lx = y$. By Lemma A.10, $x \in X_0$. □

Similarly we can prove that the Fredholm index is preserved:

Lemma A.12. *Assume that the limiting operator $\widetilde{L}\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is semi-Fredholm with index $\nu < +\infty$. Then L is semi-Fredholm with index ν both $X_{\mathrm{b}} \to Y_{\mathrm{b}}$ and $X_0 \to Y_0$.*

Proof. Consider the family of operators $L_t = \widetilde{L} + t(L - \widetilde{L})$ for $t \in [0,1]$. For each t, L_t is uniformly elliptic and oblique with limiting operator $\widetilde{L}$, so Lemmas A.8 and A.9 imply that L_t is locally proper $X_{\mathrm{b}} \to Y_{\mathrm{b}}$ and $X_0 \to Y_0$. Thus by the continuity of the index, $L_1 = L$ has the same Fredholm index as $L_0 = \widetilde{L}$, both $X_{\mathrm{b}} \to Y_{\mathrm{b}}$ and $X_0 \to Y_0$. □

Next we consider subspaces of even functions. Assume that A and B commute with the reflection operator $Su(q,p) = u(-q,p)$, and let $X^{\mathrm{e}}_{\mathrm{b}}$ denote the subspace of X_{b} consisting of functions which are even in q. Defining $Y^{\mathrm{e}}_{\mathrm{b}}$, X^{e}_0, and Y^{e}_0 similarly, the following lemma is straightforward.

Lemma A.13. *Under the above assumption, Lemmas A.8, A.9, A.10, A.12 and Corollary A.11 remain valid if we everywhere replace X_{b} by $X^{\mathrm{e}}_{\mathrm{b}}$, Y_0 by Y^{e}_0, and so on. The only exception is that in Lemmas A.8, A.9, A.10, and A.12 we must in addition require that the homogeneous limiting problem $\widetilde{L}u = 0$ have no nontrivial solutions $u \in X_{\mathrm{b}}$.*

Lastly, we state two maximum principles in general unbounded domains. First, we recall the following simple consequence of the strong maximum principle:

Lemma A.14. *Let $\Omega \subset \mathbb{R}^n$ be a domain, possibly unbounded, and suppose that*

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu$$

is a uniformly elliptic operator, with $a^{ij}, b^i, c \in C^0_{\mathrm{b}}(\overline{\Omega})$ and $c \leq 0$. If $u \in C^2_{\mathrm{b}}(\overline{\Omega})$ satisfies $u \geq 0$ on $\partial\Omega$, $Lu \leq 0$ in Ω, and

$$\liminf_{r\to\infty} \inf_{|x|=r} u(x) \geq 0, \tag{A.15}$$

then $u > 0$ in Ω unless u is constant.

Combining Lemma A.14 with the Hopf lemma, we can then easily prove the following:

Lemma A.15. *Let Ω and L be as in Lemma A.14, and let T be a (relatively open) C^1 piece of $\partial\Omega$ with outward-pointing normal ν. Suppose that $u \in C_{\mathrm{b}}^2(\overline{\Omega})$ satisfies* (A.15)*, $Lu \le 0$ in Ω, $u \ge 0$ on $\partial\Omega \setminus T$, and*

$$d^i D_i u + f u \ge 0 \qquad \text{on } T, \tag{A.16}$$

where $d^i, f \in C^0(T)$ satisfy $d^i \nu_i > 0$ and $f > 0$. Then $u > 0$ on $\Omega \cup T$ unless u is constant.

Proof. First we claim that $u \ge 0$. Suppose instead that $M = \inf_\Omega u < 0$. By the asymptotic condition (A.15) and the maximum principle Lemma A.14, u must achieve M at some point $x^* \in \partial\Omega$. Since $u \ge 0$ on $\partial\Omega \setminus T$, we must have $x^* \in T$. Since x^* is then a critical point of $u|_T$, we have $Du = \frac{\partial u}{\partial \nu}\nu$ at x^*, so that (A.16) reads

$$d^i \nu_i \frac{\partial u}{\partial \nu} + f u \ge 0 \qquad \text{at } x = x^*. \tag{A.17}$$

But the Hopf lemma implies $\frac{\partial u}{\partial \nu}(x^*) < 0$, and by assumption $d^i\nu_i, f > 0$ and $u(x^*) = M < 0$. Thus the left hand side of (A.17) is negative, a contradiction.

Since $u \ge 0$, the strong maximum principle implies $u > 0$ in Ω. It remains to show that $u > 0$ on T. Suppose that $u(x^*) = 0$ for some $x^* \in T$. Then $\inf_\Omega u = 0$, so (A.17) holds at x^*, contradicting the Hopf lemma as before. □

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