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LARGE-AMPLITUDE SOLITARY WATER WAVES WITH VORTICITY*

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Abstract. We provide the first construction of exact solitary waves of large amplitude with an arbitrary distribution of vorticity. We use continuation to construct a global connected set of symmetric solitary waves of elevation, whose profiles decrease monotonically on either side of a central crest. This generalizes the classical result of Amick and Toland.

Key words. water waves, vorticity, solitary waves

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1. Introduction.

The classical water wave problem concerns a two-dimensional, incompressible, inviscid fluid with unit density under the influence of gravity. At time t the fluid occupies the region $\{(x, y) : 0 < y < \eta(x, t)\}$ in the xy-plane; the bottom $y = 0$ is an impermeable horizontal bed, while the top $y = \eta(x, t)$ is a free surface. The velocity field (u, v) satisfies the incompressible Euler equations in the fluid domain, and the pressure P is constant on the free surface. We ignore the effect of surface tension.

We consider steady traveling waves with speed $c > 0$, for which u, v, η, P depend only $x - ct$ and y. This allows us to eliminate time t from the equations by switching to moving coordinates $(x - ct, y) \mapsto (x, y)$. In these coordinates the fluid region is

$$D_\eta = \{(x, y) \in \mathbb{R}^2 : 0 < y < \eta(x)\}.$$

Solitary waves are traveling waves satisfying the asymptotic conditions

$$\eta \to d, \quad v \to 0, \quad u \to U(y) \qquad \text{as } x \to \pm\infty,$$

uniformly in y. Here $d > 0$ is the asymptotic depth of the fluid and $U(y)$ describes the shear flow at $x = \pm\infty$.

We will work with a one-parameter family of shear flows

$$U(y) = c - FU^*(y), \tag{1.1}$$

where F is a positive dimensionless parameter and U^* is a fixed positive function, normalized so that

$$g \int_0^d \frac{dy}{U^*(y)^2} = 1, \qquad \frac{1}{F^2} = g \int_0^d \frac{dy}{(c - U(y))^2}. \tag{1.2}$$

We call U^* the *relative shear flow at infinity* and F the *generalized Froude number*. We call a wave *supercritical* if $F > 1$ and *subcritical* if $F < 1$. Local curves $\mathscr{C}_{\text{loc}}$ of small-amplitude supercritical solitary waves with F slightly bigger than 1 have been

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constructed by Ter-Krikorov [44], Hur [20], and Groves and Wahlén [18] (see section 4). In this paper we will construct large-amplitude supercritical solitary waves.

We assume that $u < c$ throughout the fluid, which in particular means there cannot be any *stagnation points* (x, y) where $(u, v) = (c, 0)$. These are points where "stagnant" fluid particles are carried along with the wave. We call a solitary wave *symmetric* if u and η are even in x, and v is odd in x. We call a symmetric wave *monotone* if in addition $\eta(x)$ is strictly decreasing for $x > 0$. We call a solitary wave *trivial* if $\eta \equiv d$, $v \equiv 0$, and $u \equiv U(y)$, and a *wave of elevation* if $\eta(x) > d$ for all $x \in \mathbb{R}$.

We define a depth $d^* \in (d, \infty]$ in terms of the shear profile U^* by

$$d^* = \int_0^d \frac{U^*(y)\,dy}{\sqrt{U^*(y)^2 - (U^*_{\min})^2}}, \qquad \text{where } U^*_{\min} = \min_{y\in[0,d]} U^*(y). \tag{1.3}$$

This is the maximum depth of a family of trivial flows considered in section 2 which generalize the one-parameter family $U = c - FU^*$. We remark that when $U^* = ay + b$ is linear so that the vorticity $\omega = Fa$ is constant, we have $d^* = \infty$ for $a = 0$ and $d^* < \infty$ for $a \neq 0$.

Our main result is the following.

THEOREM 1.1. *Fix $g, c, d > 0$, a Hölder parameter $\beta \in (0, \frac{1}{2}]$, and a strictly positive relative shear flow $U^* \in C^{2+\beta}[0, d]$ satisfying the normalization condition* (1.2)*. Then, there exists a connected set $\mathscr{C}$ of solitary waves*

$$(u, v, \eta, F) \in C^{1+\beta}(D_\eta) \times C^{1+\beta}(D_\eta) \times C^{2+\beta}(\mathbb{R}) \times (1, \infty),$$

where F determines the flow U at infinity via (1.1)*, with the following properties. $\mathscr{C}$ contains the local curve $\mathscr{C}_{\mathrm{loc}}$. Each wave $(u, v, \eta, F) \in \mathscr{C}$ is a symmetric monotone supercritical wave of elevation with $u < c$. In addition, one of the following three alternatives holds:*

(i) *(Stagnation) There is a sequence of flows $(u_n, v_n, \eta_n, F_n) \in \mathscr{C}$ and a sequence of points (x_n, y_n) such that $u_n(x_n, y_n) \nearrow c$; or*

(ii) *(Large amplitude and Froude number) There exists a sequence of flows $(u_n, v_n, \eta_n, F_n) \in \mathscr{C}$ with both $F_n \nearrow \infty$ and $\lim_{n\to\infty} \eta_n(0) \geq d^*$; or*

(iii) *(Critical wave) There exists a solitary wave of elevation (u, v, η, F) in the closure of $\mathscr{C}$ with critical Froude number $F = 1$.*

Alternative (i), stagnation, means that there are solitary waves in $\mathscr{C}$ nearly violating our assumption $u < c$. We make no claim that v is simultaneously near 0. Alternative (ii) means there are waves with an arbitrarily large Froude number and whose maximum height approaches or exceeds d^*. Note that, since waves in $\mathscr{C}$ are symmetric and monotone, $\eta(0) = \max \eta$. Finally, alternative (iii) guarantees the existence of a solitary wave of elevation with a critical Froude number. It is an open question if there exist relative shear flows U^* for which alternatives (ii) or (iii) actually occur.

As for regularity, it is known that the streamlines of each wave in $\mathscr{C}$ are analytic, except possibly for the free surface $y = \eta(x)$ [22]. Moreover, if U^* has the additional regularity $U^* \in C^{k+\beta}$ for $k \geq 3$, one can construct a continuum $\mathscr{C}$ of solutions with the additional regularity $u, v \in C^{k-1+\beta}$ and $\eta \in C^{k+\beta}$. For simplicity, in this paper we will restrict ourselves to $k = 2$.

We now specialize Theorem 1.1 to the irrotational case where $U^* = \sqrt{gd}$ is constant. Since symmetric monotone supercritical solitary waves have $F < 2$ [3], alternative (ii) cannot occur. Moreover, there are no nontrivial waves with a critical Froude number [28, 33], so alternative (iii) cannot occur either. Thus alternative (i) holds.

For irrotational and symmetric solitary waves, u is always maximized at the crest $(0, \eta(0))$, so we must have

$$u_n(0, \eta_n(0)) \to c$$

for some sequence $(u_n, v_n, \eta_n, F_n) \in \mathscr{C}$. This recovers a result of Amick and Toland, part (c) of Theorem 3.9 in [3].

Small-amplitude irrotational periodic waves were first constructed in the 1920s by Nekrasov [37] and Levi-Civita [31] using conformal mappings and power series expansions. Such conformal mappings are only available in the irrotational case. In 1934, Dubreil-Jacotin [12] devised a nonconformal coordinate transformation which permits the construction of small-amplitude periodic waves with vorticity. Subsequently, the existence of small-amplitude periodic waves has been reformulated as a bifurcation problem. This method relies heavily on compactness or Fredholm properties of the linearized operator. The periodic waves in all the above references are subcritical.

Constructing small-amplitude solitary waves is much more difficult. The domain is not bounded, and the linearized operator is nonFredholm, so we no longer have a standard bifurcation problem. Solutions were constructed as long-wavelength limits of periodic waves [30, 43] and using an iteration method [15]. Beale [4] used a Nash–Moser implicit function theorem, and Mielke [36] used spatial dynamics methods, reformulating the water wave problem as an evolution equation with the horizontal variable x playing the role of time and performing a center-manifold type reduction to a two-dimensional equation. All of these constructions involve some sort of rescaling of the horizontal variable x.

As in the periodic case, the presence of vorticity complicates the construction of small-amplitude solitary waves, in particular by preventing the use of conformal mappings. For formal results see [5, 7, 14]. The first rigorous construction is due to Ter-Krikorov [44]. Later Hur [20] generalized the methods of [4], and shortly thereafter Groves and Wahlén [18] gave an alternate proof using spatial dynamics methods. The solitary waves in all of the above references (rotational and irrotational) are supercritical. Although irrotational symmetric monotone waves of elevation are necessarily supercritical [33], this is not known in general.

We will construct large-amplitude solitary waves with vorticity by continuing from waves with small amplitude. This construction, however, requires more information about small-amplitude waves than is given in [20] or [18]. Most importantly, we need to show that certain linearized operators are invertible. Compared with [20], [18] gives a more detailed description of the solutions, identifying them with the homoclinic orbits of a two-dimensional reduced equation. In order to prove invertibility, we linearize each step of the reduction in [18], analyze the linearization of the reduced equation, and reverse the various changes of variable. We also show that these small-amplitude solitary waves are the unique such waves with nearly critical Froude number.

Large-amplitude irrotational periodic waves were first constructed by Krasovskiĭ [29]. Keady and Norbury [24] later used the global bifurcation theory of Rabinowitz [41] to obtain a connected set of solutions. Toland [45] and McLeod [34] showed that this continuum of solutions contained a wave with a stagnation point at its crest, proving the celebrated Stokes conjecture [42]. In the case of vorticity, Constantin and Strauss [9] constructed large-amplitude waves including a sequence of waves approaching stagnation in that $\sup u_n \nearrow c$. While the maximum value of u must occur at the crest for irrotational waves [45], numerical evidence implies that this is not always the case with vorticity [27]. Because of the vorticity, Constantin and Strauss cannot reduce the water wave problem to an integral equation on the boundary as in all the irrotational papers. Instead, they apply the Dubreil-Jacotin transformation, obtaining a nonlinear elliptic boundary value problem with a fully nonlinear boundary condition. In order to extend a local curve of small-amplitude solutions, they use global bifurcation theory, which is based on topological degree arguments. Because of the nonlinear boundary condition, they use a degree developed by Healey and Simpson [19] instead of the usual Leray–Schauder degree.

Large-amplitude solitary waves are much more difficult to construct than large-amplitude periodic waves. As with the small-amplitude problem, this is due to the unboundedness of the domain and the non-Fredholmness of the linearized operator. In addition to preventing the use of a Lyapunov–Schmidt argument, this singular behavior is an obstruction to defining a topological degree. The construction of large-amplitude irrotational solitary waves is due to Amick and Toland [2, 3]. In order to get around the above obstruction, they apply the usual global bifurcation theory to a sequence of approximate problems. Taking weak limits, they then construct a connected set of solitary waves, including waves which are arbitrarily close to stagnation at their crests. In [2], the approximate problems are periodic water wave problems with increasing wavelengths. Both papers make use of conformal mappings to reduce the solitary water wave problem to a Nekrasov-type integral equation on the free surface. Large-amplitude solitary waves are also constructed in [6]. Until now, there has been no existence theory for large-amplitude solitary water waves with vorticity. The problem can no longer be reduced to an integral equation on the free surface, and the method of approximate problems seems not to work.

Our approach to constructing large-amplitude waves is quite different from [2, 3] and does not involve approximate problems. As in [9], the main ingredient is the topological degree. In order to avoid the singular behavior at $F = 1$, we work with waves whose Froude number is uniformly supercritical, say $F > 1 + \delta$ for some small parameter $\delta > 0$. This restriction is helpful because the linearized operators with $F > 1$ are Fredholm with index 0. We also need to verify a crucial compactness condition called local properness, and for this we work in weighted Hölder spaces and use results of Volpert and Volpert [48] for general elliptic problems in unbounded domains. Because the degree is only defined for $F > 1$, we need an alternate theory for small-amplitude waves, and this is where we use the methods and results of [18]. For δ sufficiently small, we first find a nontrivial solitary wave with $F > 1 + \delta$ whose associated linearized operator is invertible. Then we use our topological degree together with a continuation argument in the spirit of Rabinowitz (see [26] and [41]) to obtain a global continuum of solutions with $F > 1 + \delta$. Theorem 1.1 is finally proved by sending $\delta \to 0$ and analyzing the various alternatives.

In section 1.1, we perform the Dubreil-Jacotin transformation, which, under our no-stagnation assumption $u < c$, transforms the solitary water wave problem into an elliptic boundary value problem for a function $w(x, s)$ in the infinite strip $\Omega = \mathbb{R} \times (0, 1)$. We use the divergence formulation first introduced in [10], and the dimensionless variables from [18]. The equation is quasilinear with a fully nonlinear boundary condition on the upper boundary of Ω. Using these variables, we define the global continuum $\mathscr{C}$, making precise the sense in which it is connected. We also state Theorem 1.3, which is a more precise version of Theorem 1.1.

In section 2, we derive several properties of solitary waves with $u < c$. First, we introduce a standard family of trivial flows. The maximum depth of these flows is the depth $d^* > d$ appearing in Theorem 1.1 and defined in (1.3). Using a maximum principle argument to compare solitary waves to these trivial flows, we show that all nontrivial solitary waves with $F \geq 1$ are waves of elevation. This answers a question raised in [21] and implies that all nontrivial supercritical solitary waves are monotone symmetric waves of elevation. The converse, that symmetric and monotone waves of elevation are supercritical, is known in the irrotational case [33] but not in general. A similar maximum principle argument shows that F can be bounded above by a constant depending only on U^* and the maximum height $\max \eta$, provided $d < \max \eta < d^*$. Similar bounds and elevation, symmetry, and monotonicity properties were shown by Craig and Sternberg in the irrotational case [11]. Finally, we prove an equidecay property for certain families of supercritical solitary waves, which is new even for irrotational waves. We use an invariant sometimes called the flow force together with the above monotonicity result, a lower bound on the pressure [46], and a translation argument inspired by [39]. Similar arguments may be useful in studying monotone solutions to other problems in infinite cylinders.

In section 3, we formulate the solitary water wave problem as a nonlinear operator equation. When the Froude number is uniformly bounded away from 1 and $+\infty$, we show that this nonlinear operator has all of the properties necessary to define the topological degree, which we will do in section 5. This is more subtle than in the periodic case because of the unbounded domain, which causes a loss of compactness. Though the necessary lemmas are nonstandard, they are relatively straightforward to prove, depending essentially only on the domain, ellipticity, and the divergence structure of the equation. Since we will need similar results again in section 5, we defer many of these lemmas to Appendix A. In section 3.2, we show that the linearized operators are Fredholm of index 0 when $F > 1$, and we analyze their spectral properties in section 3.3. In section 3.4, we show that the nonlinear operator satisfies a compactness condition called local properness. Here we use an argument from [48] that requires a weight σ as $x \to \pm\infty$. The weight function σ is assumed to be smooth, to have $\sigma \to \infty$ as $x \to \pm\infty$, and to satisfy a subexponential growth condition, but is otherwise arbitrary. It is worth emphasizing that section 3.4 is the only place in the paper where weights are truly essential. The weight function σ is left arbitrary in the bulk of the paper but is eventually fixed in sections 5.6 and 5.7.

In section 4, we study small-amplitude solitary waves using the methods and results of [18]. Our main result is that the operators obtained by linearizing about these solutions are invertible. In section 4.1, we perform the various changes of variable which transform the water wave problem into an evolution equation with x playing the role of time. This is the only place where the assumption $\beta \leq 1/2$ in Theorem 1.1 is convenient. In section 4.2, we consider the linearized problem and prove an exponential-dichotomy type result. In section 4.3, we exhibit the construction of a two-dimensional center manifold containing all small bounded solutions and consider the reduced two-dimensional equation on this manifold. This is a reduction analogous to the Lyapunov–Schmidt method for bifurcation problems. In each of the above steps we need a more detailed description than is provided by [18], and in particular more information concerning the various linearized problems. Combining this information with an elementary fact about homoclinic orbits of two-dimensional equations, we prove the desired invertibility in section 4.4. Finally, we use the reduced system together with the elevation result from section 2 to conclude that there is a unique small-amplitude solitary wave for each Froude number slightly greater than 1.

Section 5 is devoted to the proof of Theorem 1.1. In section 5.1, we define a weighted continuum $\mathscr{C}_\sigma^\delta \subset \mathscr{C}$ of solutions depending on a weight function σ to be chosen later and a small parameter $\delta > 0$. In section 5.2, we use the invertibility results of section 3 and 4 to analyze the connectedness properties of $\mathscr{C}$ and $\mathscr{C}_\sigma^\delta$. In section 5.3, we define the Healey–Simpson degree for the nonlinear operator introduced in section 3. In section 5.4, we apply a degree-theoretic "global implicit function theorem" [26] near one of the small-amplitude solutions in $\mathscr{C}_{\mathrm{loc}}$, again using the invertibility from section 4. We are left with several possibilities for the weighted continuum $\mathscr{C}_\sigma^\delta$: either it is unbounded, or it contains new waves whose Froude number F is δ-close to 1 or $+\infty$. If $\mathscr{C}_\sigma^\delta$ is unbounded, then, letting w be the function defined in section 1.1, there is a sequence in $\mathscr{C}_\sigma^\delta$ with $\|\sigma w_n\|_{C^{2+\beta}} \to \infty$. In section 5.5, we reduce this condition to one involving fewer partial derivatives. This is done by combining regularity results for fully nonlinear elliptic problems due to Lieberman [32] with the weighted Schauder estimates from Appendix A and the lower bound on the pressure from section 2. In sections 5.6 and 5.7, we assume that alternative (i) of Theorem 1.1, stagnation, does not hold, and apply the equidecay result from section 2 to construct a weight function σ for which $\|\sigma w\|_{C^{2+\beta}}$ is uniformly bounded along the unweighted continuum $\mathscr{C}$. We then send $\delta \to 0$ and address the remaining possibilities, that the Froude number F might approach 1 or $+\infty$. For large F, we apply the lower bound on the maximum height from section 2 to obtain alternative (ii), while for F near 1 the uniqueness result from section 4 leads to alternative (iii).

Appendix A is a collection of lemmas on linear and nonlinear elliptic problems in unbounded domains (infinite strips in particular) which are used throughout the paper. We prove Schauder-type estimates as well as local properness and invertibility properties in both weighted and unweighted Hölder spaces. To prove local properness, we use ideas from [48], which considers general elliptic systems in general unbounded domains. In particular, we introduce the notion of so-called limiting problems obtained by sending the horizontal variable $x \to \pm\infty$ in the coefficients. For the reader's convenience, we provide greatly simplified proofs of some results in [48] in our more restricted setting.

1.1. Reformulation. Let $\Omega \subset \mathbb{R}^n$ be a domain, possibly unbounded. We say that $\varphi \in C_{\mathrm{c}}^\infty(\Omega)$ if $\varphi \in C^\infty(\Omega)$ and the support of φ is a compact subset of Ω. Similarly $\varphi \in C_{\mathrm{c}}^\infty(\overline{\Omega})$ if $\varphi \in C^\infty(\overline{\Omega})$ and the support of φ is a compact subset of $\overline{\Omega}$. For $k \in \mathbb{N}$ and $\beta \in [0,1)$, we denote the $C^{k+\beta}$ Hölder norm of a function u on Ω by $|u|_{k+\beta;\Omega}$. When Ω is clear from context, we will simply write $|u|_{k+\beta}$. We say that $u \in C^{k+\beta}(\Omega)$ if $|\varphi u|_{k+\beta} < \infty$ for all $\varphi \in C_{\mathrm{c}}^\infty(\Omega)$, $u \in C^{k+\beta}(\overline{\Omega})$ if $|\varphi u|_{k+\beta} < \infty$ for all $\varphi \in C_{\mathrm{c}}^\infty(\overline{\Omega})$, and $u \in C_{\mathrm{b}}^{k+\beta}(\overline{\Omega})$ if $|u|_{k+\beta} < \infty$. We say that $u_n \to u$ in $C_{\mathrm{loc}}^{k+\beta}(\overline{\Omega})$ if $|\varphi(u_n - u)|_{k+\beta} \to 0$ for all $\varphi \in C_{\mathrm{c}}^\infty(\overline{\Omega})$.

Having eliminated time t through the change of variables $(x - ct, y) \mapsto (x, y)$, the velocity field (u, v) satisfies

$$(u-c)u_x + vu_y = -P_x, \qquad (u-c)v_x + vv_y = -P_y - g, \qquad u_x + v_y = 0 \tag{1.4a}$$

in the fluid domain D_η together with the boundary conditions

$$v = 0 \ \text{ on } y = 0, \qquad v = (u-c)\eta_x \ \text{ on } y = \eta(x), \qquad P = P_{\mathrm{atm}} \ \text{ on } y = \eta(x) \tag{1.4b}$$

and the asymptotic conditions

$$\eta \to d, \quad v \to 0, \quad u \to U(y) = c - FU^*(y) \qquad \text{as } x \to \pm\infty, \tag{1.4c}$$

uniformly in y. Here P_{atm} is the (constant) atmospheric pressure, $g > 0$ is the gravitational constant of acceleration, d is the asymptotic depth, $c > 0$ is the wave speed, $U^* > 0$ is the relative shear flow at infinity, and $F > 0$ is the generalized Froude number.

Our first step is to eliminate the pressure by introducing the (relative) stream function ψ, defined by

$$\psi_x = -v, \qquad \psi_y = u - c, \qquad \psi(x, 0) = 0.$$

The assumption $u - c = \psi_y < 0$ guarantees [9] that

$$\omega = v_x - u_y = -\Delta\psi = \gamma(\psi)$$

for some function γ called the *vorticity function.*

In terms of ψ, γ, and η, (1.4) can then be rewritten as

$$\begin{cases} \Delta\psi = -\gamma(\psi) & \text{in } 0 < y < \eta(x), \\ \psi = 0 & \text{on } y = 0, \\ \psi = m & \text{on } y = \eta(x), \\ \tfrac{1}{2}|\nabla\psi|^2 + g(\eta - d) = \tfrac{\lambda}{2} & \text{on } y = \eta(x), \end{cases} \tag{1.5a}$$

together with the asymptotic conditions

$$\eta \to d, \quad \psi_x \to 0, \quad \psi_y \to -FU^*(y) \qquad \text{as } x \to \pm\infty, \tag{1.5b}$$

uniformly in y. Here (as can been seem from (1.5b)) the flux $m < 0$ and Bernoulli constant λ appearing in (1.5a) are given in terms of the relative shear flow U^* at infinity and Froude number F by

$$m = -F\int_0^d U^*(y)\,dy, \qquad \lambda = (FU^*(d))^2, \tag{1.6}$$

and the vorticity function γ is given in terms of U^* and F by

$$\gamma(-s) = FU^*_y(y), \qquad \text{where } \ s = F\int_0^y U^*\,dy'. \tag{1.7}$$

This last definition (1.7) makes use of the fact that s is strictly increasing as a function of y, running from 0 to $-m$.

Following [18], we define the dimensionless variables

$$(\tilde{x}, \tilde{y}) = \frac{1}{d}(x, y), \quad \tilde{\eta}(\tilde{x}) = \frac{1}{d}\eta(x), \quad \tilde{\psi}(\tilde{x}, \tilde{y}) = \frac{1}{|m|}\psi(x, y), \quad \tilde{\gamma}(\tilde{\psi}) = \frac{d^2}{|m|}\gamma(\psi), \tag{1.8}$$

where we have rescaled lengths by d and velocities by $|m|/d$. In these variables (1.5a) becomes

$$\begin{cases} \Delta\tilde{\psi} = -\tilde{\gamma}(\tilde{\psi}) & \text{in } 0 < \tilde{y} < \tilde{\eta}(\tilde{x}), \\ \tilde{\psi} = 0 & \text{on } \tilde{y} = 0, \\ \tilde{\psi} = -1 & \text{on } \tilde{y} = \tilde{\eta}(\tilde{x}), \\ \dfrac{1}{2}|\nabla\tilde{\psi}|^2 + \alpha(\tilde{\eta} - 1) = \dfrac{\mu}{2} & \text{on } \tilde{y} = \tilde{\eta}(\tilde{x}), \end{cases} \tag{1.9a}$$

and the asymptotic condition (1.5b) becomes

$$\tilde{\psi}_{\tilde{x}} \to 0, \qquad \tilde{\eta} \to 1, \qquad \tilde{\psi}_{\tilde{y}}(\tilde{x}, \tilde{y}) \to -\frac{U^*(\tilde{y}d)d}{\int_0^d U^*\,dy} \qquad \text{as } \tilde{x} \to \pm\infty, \tag{1.9b}$$

uniformly in $\tilde{y}$, where α is the rescaled gravity and μ is the rescaled Bernoulli constant,

$$\alpha = \frac{gd^3}{m^2} = \frac{gd^3}{F^2}\left(\int_0^d U^*\,dy\right)^{-2}, \tag{1.10}$$

$$\mu = \frac{\lambda d^2}{m^2} = d^2(U^*(d))^2\left(\int_0^d U^*\,dy\right)^{-2}.$$

The asymptotic condition (1.9b) and $\tilde{\psi}(\tilde{x},0)=0$ imply

$$\tilde{\psi}(\tilde{x},\tilde{y}) \to \tilde{\Psi}(\tilde{y}) := -\frac{\int_0^{d\tilde{y}} U^*(y)\,dy}{\int_0^d U^*(y)\,dy} \qquad \text{as } x \to \pm\infty,$$

uniformly in $\tilde{y}$.

The critical value $\alpha = \alpha_{\mathrm{cr}}$ corresponding to $F=1$ is given by

$$\alpha_{\mathrm{cr}} = gd^3\left(\int_0^d U^*\,dy\right)^{-2} \tag{1.11}$$

with $\alpha < 1$ for $F > 1$ and $\alpha > 1$ for $F < 1$. The dimensionless vorticity function $\tilde{\gamma}$ is given implicitly in terms of the relative shear flow U^* at infinity by

$$\tilde{\gamma}(-\tilde{s}) = \frac{d^2 U_y^*(y)}{\int_0^d U^*\,dy}, \qquad \text{where } \tilde{s} = \frac{\int_0^y U^*\,dy}{\int_0^d U^*\,dy}. \tag{1.12}$$

In Theorem 1.1, g, d, U^* are fixed while F is a parameter. Looking at (1.10)–(1.12), we see that $\mu, \tilde{\gamma}$ are fixed, while α is proportional to $1/F^2$. This is an advantage over the original dimensional variables, where λ and γ both depended on F.

We next apply the Dubreil-Jacotin transformation [12]. Setting

$$\tilde{s} = -\tilde{\psi}(\tilde{x},\tilde{y}), \qquad \tilde{h} = \tilde{y}, \tag{1.13}$$

we treat $(\tilde{x},\tilde{s}) \in \mathbb{R}\times(0,1)$ as independent variables and $\tilde{h}(\tilde{x},\tilde{s})$ as the dependent variable, transforming the domain of the problem into the (fixed) infinite strip $\Omega = \mathbb{R}\times(0,1)$. In these new variables (1.5) is equivalent to [9]

$$\left(\frac{\tilde{h}_{\tilde{x}}}{\tilde{h}_{\tilde{s}}}\right)_{\tilde{x}} - \left(\frac{1+\tilde{h}_{\tilde{x}}^2}{2\tilde{h}_{\tilde{s}}^2}\right)_{\tilde{s}} + \tilde{\gamma}(-\tilde{s}) = 0 \qquad 0 < \tilde{s} < 1, \tag{1.14a}$$

$$\frac{1+\tilde{h}_{\tilde{x}}^2}{2\tilde{h}_{\tilde{s}}^2} + \alpha(\tilde{h}-1) = \frac{\mu}{2} \qquad \tilde{s} = 1, \tag{1.14b}$$

$$\tilde{h} = 0 \qquad \tilde{s} = 0, \tag{1.14c}$$

together with the asymptotic condition

$$\tilde{h}(\tilde{x},\tilde{s}) \to \tilde{H}(\tilde{s}), \quad \tilde{h}_{\tilde{x}} \to 0, \quad \tilde{h}_{\tilde{s}} \to \tilde{H}_{\tilde{s}} \qquad \text{as } \tilde{x} \to \pm\infty, \tag{1.14d}$$

uniformly in $\tilde{s}$. The asymptotic height function $\tilde{H}$ is the solution of the differential equation

$$\tilde{H}_{\tilde{s}}(\tilde{s}) = -\frac{1}{\tilde{\Psi}_{\tilde{y}}(\tilde{H}(\tilde{s}))} = \frac{\int_0^d U^*\,dy}{U^*(\tilde{H}(\tilde{s})d)d}, \qquad \tilde{H}(0) = 0,$$

and satisfies $\tilde{H}(1) = 1$. This can be seen, for instance, from the equivalent definition

$$\int_0^{\tilde{H}(\tilde{s})d} U^*\, dy = \tilde{s} \int_0^d U^*\, dy. \tag{1.15}$$

The divergence form of (1.14) first appeared in [10]. We emphasize that the physical shear flow $U(y) = c - FU^*(y)$ represented by $\tilde{H}$ depends on the Froude number F, even though the formula (1.15) does not. The dimensionless height function $\tilde{h}$ is related to the original variables u, v, m, d by

(1.16)
$$\tilde{h}_{\tilde{s}}(\tilde{x}, \tilde{s}) = -\frac{1}{\tilde{\psi}_{\tilde{y}}(\tilde{x}, \tilde{y})} = \frac{|m|}{d} \frac{1}{c - u(x,y)}, \qquad \tilde{h}_{\tilde{x}}(\tilde{x}, \tilde{s}) = -\frac{\tilde{\psi}_{\tilde{x}}(\tilde{x}, \tilde{s})}{\tilde{\psi}_{\tilde{y}}(\tilde{x}, \tilde{y})} = \frac{v(x,y)}{u(x,y) - c}.$$

In particular, our assumption $u - c = \psi_y < 0$ is equivalent to $\tilde{h}_{\tilde{s}} > 0$, so the quotients appearing in (1.14) are well defined. For $\tilde{h} \in C^2_{\mathrm{b}}(\overline{\Omega})$, we will show in section 3.1 that $\inf_\Omega \tilde{h}_{\tilde{s}} > 0$ implies that (1.14a) is uniformly elliptic and that the boundary condition (1.14b) is uniformly oblique. This is a major advantage of the divergence formulation [10] over the nondivergence formulation, in which an extra condition is needed to ensure obliqueness [9].

To simplify notation we will from now on *drop the tildes* on dimensionless variables. Since we will often be interested in the rates of decay in (1.14d), we define

$$w(x, s) := h(x, s) - H(s) \tag{1.17}$$

and work with w instead of h. Similarly, since small-amplitude waves have α close to α_{cr} and less than α_{cr}, we work with

$$\zeta := \alpha_{\mathrm{cr}} - \alpha. \tag{1.18}$$

Thus ζ *is positive for supercritical waves and negative for subcritical waves*, and the small-amplitude waves constructed in [18, 20] have ζ small and positive. Since $\alpha > 0$ (see its definition (1.10)), we will always assume $\zeta < \alpha_{\mathrm{cr}}$.

We ultimately formulate the solitary water wave problem in terms of (ζ, w). The nonlinear equations (1.14a) and (1.14b) become

$$\left(\frac{w_x}{H_s + w_s}\right)_x + \left(-\frac{1 + w_x^2}{2(H_s + w_s)^2}\right)_s + \gamma(-s) = 0 \qquad 0 < s < 1, \tag{1.19a}$$

$$\frac{1 + w_x^2}{2(H_s + w_s)^2} + (\alpha_{\mathrm{cr}} - \zeta) w = \frac{\mu}{2} \qquad s = 1. \tag{1.19b}$$

The remaining conditions that we place on (ζ, w) are

$$\zeta < \alpha_{\mathrm{cr}}, \tag{1.19c}$$
$$w = 0 \text{ on } s = 0, \tag{1.19d}$$
$$w \in C^{2+\beta}_{\mathrm{b}}(\overline{\Omega}), \tag{1.19e}$$
$$w,\ Dw,\ D^2 w \to 0, \text{ as } x \to \pm\infty, \tag{1.19f}$$
$$\inf_\Omega (H_s + w_s) > 0, \tag{1.19g}$$
$$w \text{ is even in } x. \tag{1.19h}$$

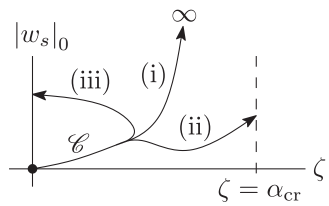

FIG. 1. *The three alternatives in Theorem* 1.3.

The first condition (1.19c) enforces the positivity of α (defined in (1.10)), while (1.19d) is (1.14c). The asymptotic condition (1.19f) is a stronger version of (1.14d) also involving second derivatives, (1.19g) enforces $h_s > 0$, and (1.19h) enforces symmetry.

From now on, we will refer to a pair (ζ, w) satisfying (1.19) as a solitary wave. We call (ζ, w) supercritical if $\zeta > 0$, trivial if $w \equiv 0$, a wave of elevation if $w(x, 1) > 0$ for all $x \in \mathbb{R}$, and monotone if $w_x < 0$ for $x > 0$. We will see in the proof of Proposition 1.4 below that this terminology is consistent with our earlier definitions in terms of (u, v, η, F).

DEFINITION 1.2 (global continuum). *The set $\mathscr{S}$ of supercritical waves is*

$$\mathscr{S} = \big\{(\zeta, w) : (\zeta, w) \text{ satisfies (1.19)}, \ 0 < \zeta < \alpha_{\rm cr}\big\},$$

which we view as a subset of $\mathbb{R} \times C^{2+\beta}_{\rm b}(\overline{\Omega})$. The global continuum $\mathscr{C}$ is the connected component of $\mathscr{S}$ in $\mathbb{R} \times C^{2+\beta}_{\rm b}(\overline{\Omega})$ containing the local curve $\mathscr{C}_{\rm loc}$ of small-amplitude solutions.

We note that $\mathscr{S}$ contains trivial solutions $(\zeta, 0)$ with $\zeta > 0$. We will show in section 5.2 that $\mathscr{C}$ contains only nontrivial solutions $w \not\equiv 0$.

We now show that Theorem 1.1 is implied by the following theorem in the (ζ, w) variables, whose alternatives are illustrated in Figure 1.

THEOREM 1.3. *Fix $g, c, d > 0$, a Hölder parameter $\beta \in (0, \frac{1}{2}]$, and a strictly positive relative shear flow $U^* \in C^{2+\beta}[0, d]$ satisfying the normalization condition* (1.2)*. Defining the global continuum $\mathscr{C}$ as above, all solutions $(\zeta, w) \in \mathscr{C}$ satisfy $w(x, 1) > 0$ for $x \in \mathbb{R}$ as well as $w_x < 0$ for $x > 0$ and $0 < s \leq 1$. In addition, one of the following three alternatives holds:*

(i) $\sup_{(\zeta,w)\in\mathscr{C}} |w_s|_0 = \infty$;
(ii) $\sup_{(\zeta,w)\in\mathscr{C}} \zeta = \alpha_{\rm cr}$; *or*
(iii) *there exists a solution $(0, w)$ in the closure of $\mathscr{C}$ with $w(x, 1) > 0$ for $x \in \mathbb{R}$.*

If alternative (ii) *holds, then there exists a sequence $(\zeta_n, w_n) \in \mathscr{C}$ with both $\zeta_n \to \alpha_{\rm cr}$ and*

$$\lim_{n\to\infty} w_n(0, 1) \geq d^*/d - 1.$$

PROPOSITION 1.4. *Theorem* 1.3 *implies Theorem* 1.1*.*

Proof. We reintroduce the notation (1.8) to differentiate between dimensionless and dimensional versions of x, y, η, h, ψ. Recall that lengths are rescaled by d and velocities by $|m|/d$. The proof that solutions (ζ, w) of (1.19) yield solutions (u, v, η, F) of (1.4) with $u < c$ is nearly identical to the one found in [9] and is omitted.

Suppose that $(\zeta, w) \in \mathscr{C}$ corresponds to a solitary wave (u, v, η, F). Combining (1.10), (1.11), and (1.18), we get

$$F = \left(\frac{\alpha_{\rm cr}}{\alpha_{\rm cr} - \zeta}\right)^{1/2}, \tag{1.20}$$

where we recall that $\alpha_{\mathrm{cr}} > 0$ is fixed and given in (1.11). Thus the condition $0 < \zeta < \alpha_{\mathrm{cr}}$ appearing in the definition of $\mathscr{S}$ (and hence the definition of $\mathscr{C}$) is equivalent to supercriticality $1 < F < \infty$. Next we use (1.13) and (1.17) together with the scaling (1.8) to get

$$\eta(x) = d(1 + w(\tilde{x}, 1)). \tag{1.21}$$

Thus $w(\tilde{x}, 1) > 0$ for all $x \in \mathbb{R}$ is equivalent to (u, v, η, F) being a wave of elevation, $\eta(x) > d$ for $x \in \mathbb{R}$. Similarly, $w_x < 0$ for $x > 0$ and $0 < s \leq 1$ implies the monotonicity of (u, v, η, F), $\eta_x < 0$ for $x > 0$.

Combining (1.16) (which follows from (1.13)) and the scaling (1.8), we find

$$\begin{aligned} \frac{d}{|m|}(c - u(x,y)) &= \frac{1}{\tilde{h}_{\tilde{s}}(\tilde{x}, \tilde{s})} = \frac{1}{\tilde{H}_{\tilde{s}}(\tilde{s}) + w_{\tilde{s}}(\tilde{x}, \tilde{s})}, \\ \frac{d}{|m|}v(x,y) &= \frac{\tilde{h}_{\tilde{x}}(\tilde{x}, \tilde{s})}{\tilde{h}_{\tilde{s}}(\tilde{x}, \tilde{s})} = \frac{w_{\tilde{x}}(\tilde{x}, \tilde{s})}{\tilde{H}_{\tilde{s}}(\tilde{s}) + w_{\tilde{s}}(\tilde{x}, \tilde{s})}. \end{aligned} \tag{1.22}$$

Thus the evenness of w in $\tilde{x}$, (1.19h), is equivalent to the symmetry of (u, v, η, F). Using the definition (1.6) of m, we find

$$\frac{d}{|m|} = \frac{d}{F\int_0^d U^*(y)\,dy} =: \frac{C_1}{F}$$

for some positive constant C_1. Thus (1.22) can be rewritten

$$\frac{C_1}{F}(c - u(x,y)) = \frac{1}{\tilde{H}_{\tilde{s}}(\tilde{s}) + w_{\tilde{s}}(\tilde{x}, \tilde{s})}. \tag{1.23}$$

Assume that alternative (ii) holds in Theorem 1.3. Then there exists a sequence $(\zeta_n, w_n) \in \mathscr{C}$ with $\zeta_n \nearrow \alpha_{\mathrm{cr}}$ and $\lim_{n\to\infty} w_n(0,1) \geq d^*/d - 1$. Letting (u_n, v_n, η_n, F_n) be the corresponding solitary waves, we have from (1.20) that $F_n \to \infty$, and from (1.21) that

$$\lim_{n\to\infty} \eta_n(0) = d\left(1 + \lim_{n\to\infty} w_n(0,1)\right) \geq d^*.$$

Thus alternative (ii) holds in Theorem 1.1.

Now suppose that alternative (iii) holds in Theorem 1.3. Then there exists a solution $(0, w)$ in the closure of $\mathscr{C}$ with $w(x, 1) > 0$ for all $x \in \mathbb{R}$. From (1.20) we see that the corresponding wave (u, v, η, F) has $F = 1$, and from (1.21) we see that $\eta(x) > d$ for all $x \in \mathbb{R}$. Thus alternative (iii) of Theorem 1.1 holds.

Finally, suppose in Theorem 1.3 that alternative (i) holds while alternative (ii) does not hold. Then $\sup_{(\zeta,w)\in\mathscr{C}} \zeta < \alpha_{\mathrm{cr}}$. By (1.20), solitary waves (u, v, η, F) corresponding to $(\zeta, w) \in \mathscr{C}$ have $F > \delta$ for some uniform $\delta > 0$. Now we use (1.23). Since $\tilde{H}$ is fixed, alternative (i) in Theorem 1.3 means that there exists a sequence of waves (u_n, v_n, η_n, F_n) in $\mathscr{C}$ and points (x_n, y_n) with

$$\frac{C_1}{F_n}(c - u_n(x_n, y_n)) \to 0.$$

But $F_n > \delta$ for each n, so this can only happen if $u_n(x_n, y_n) \to c$, which is alternative (i) of Theorem 1.1. $\square$

In the remainder of the paper we eliminate the shear profile U^* in favor of the vorticity function γ. It is customary [9, 18, 20] to define

$$\Gamma(s) = -\int_s^1 \gamma(-t)\,dt, \qquad a(s) = \sqrt{\mu + 2\Gamma(s)}. \tag{1.24}$$

The constant μ is then the unique solution to

$$\int_0^1 \frac{ds}{\sqrt{\mu + 2\Gamma}} = 1, \tag{1.25}$$

and the critical value α_{cr} of α and asymptotic height function H are given by

$$\frac{1}{\alpha_{\mathrm{cr}}} = \int_0^1 a(s)^{-3}\,ds, \qquad H(s) = \int_0^s a(t)^{-1}\,dt. \tag{1.26}$$

Finally, the depth d^* defined in (1.3) is given by

$$d^* = d\int_0^1 \frac{ds}{\sqrt{2\Gamma(s) - 2\Gamma_{\min}}}, \qquad \Gamma_{\min} = \min_{s\in[0,1]} \Gamma(s). \tag{1.27}$$

The formulas (1.24)–(1.27) can be derived from our earlier definitions. In particular, while the existence of μ satisfying (1.25) places a restriction on γ, it does not involve any additional restrictions on the relative shear flow U^*. We observe that the regularity $U^* \in C^{2+\beta}[0,d]$ assumed in Theorem 1.1 implies

$$\gamma \in C^{1+\beta}[-1,0], \qquad \Gamma, a \in C^{2+\beta}[0,1], \qquad H \in C^{3+\beta}[0,1].$$

2. Elevation, bounds, and decay. This section is devoted to the proofs of the following five propositions. See (1.20) and (1.21) for the relationship between w, η, F, and ζ.

Proposition 2.1 (elevation). *Every nontrivial solitary wave with $F \geq 1$, not necessarily symmetric, is a wave of elevation. More precisely, if (ζ, w) is a nontrivial solution of* (1.19a)–(1.19g) *with $\zeta \geq 0$, then $w(x,1) > 0$ for all $x \in \mathbb{R}$, and $w > 0$ in Ω.*

Proposition 2.2 (symmetry and monotonicity). *Every nontrivial supercritical solitary wave is symmetric and monotone. More precisely, if (ζ, w) is a solution of* (1.19a)–(1.19g) *with $\zeta > 0$ and $w \not\equiv 0$, then, after a translation in x, w is even in x. Moreover, $w_x < 0$ for $x > 0$ and $0 < s \leq 1$.*

Proposition 2.3 (upper bound on Froude number). *If the maximum height $\max\eta$ of a solitary wave, not necessarily supercritical, satisfies $d < \max\eta < d^* \leq \infty$, then the Froude number F is bounded above by a constant C depending only on U^* and $\max\eta$. More precisely, let (ζ, w) be a nontrivial solution of* (1.19a)–(1.19g) *with no sign condition on ζ. If $d^* < \infty$ and $\max w(x,1) \leq d^*/d - 1$, then $\alpha_{\mathrm{cr}} - \zeta > C$, where the constant $C > 0$ is independent of (ζ, w). If $d^* = \infty$ and $\max w(x,1) < M < \infty$, then $\alpha_{\mathrm{cr}} - \zeta > C$, where the constant $C > 0$ depends only on M.*

Proposition 2.4 (bounds on first derivatives). *Let (ζ, w) be a solitary wave with $\zeta \geq 0$. Then there exist constants $\delta_*, M > 0$ depending only on γ so that*

$$\inf_\Omega (H_s + w_s) \geq \delta_*, \qquad |w_x|_0 \leq M(1 + |w_s|_0).$$

PROPOSITION 2.5 (equidecay at infinity). *Let $\mathscr{W}$ be any collection of supercritical solitary waves (ζ, w) for which*

$$\sup_{(\zeta,w)\in\mathscr{W}} |w|_{2+\beta} < \infty.$$

Then $\mathscr{W}$ has the equidecay property

$$\lim_{x\to\pm\infty} \sup_{(\zeta,w)\in\mathscr{W}} \sup_{s\in[0,1]} |w(x,s)| = 0.$$

Propositions 2.1–2.3 are proved in the irrotational case by Craig and Sternberg [11] but are new for waves with an arbitrary distribution of vorticity. Proposition 2.2 follows from Proposition 2.1 and a theorem from [21], and Proposition 2.4 is a consequence of a lower bound on the pressure [46]. Proposition 2.5, on the other hand, is new even in the irrotational case.

In section 2.1, we will introduce a family $H(s;\nu)$ of trivial flows, that is, x-independent solutions of (1.14a) and (1.14c). We will then prove Propositions 2.1 and 2.3 in section 2.2 by applying maximum principle arguments to $h - H(s;\nu)$ for various values of ν. In section 2.3, we will prove Proposition 2.4 using a maximum principle argument [46] involving the pressure. Finally, in section 2.4 we will prove Proposition 2.5 using Propositions 2.2–2.4 and a translation argument.

2.1. Trivial flows. In this section we are interested in solutions h of (1.14a) and (1.14c) which are independent of x. These represent horizontal laminar flows with constant depth and are solutions of

$$\left(-\frac{1}{2h_s(s)^2}\right)' + \gamma(-s) = 0, \qquad h(0) = 0. \tag{2.1}$$

All solutions of (2.1) with $h_s(s) > 0$ on $[0,1]$ are of the form

$$h(s) = H(s;\nu) := \int_0^s a(t;\nu)^{-1}\,dt, \qquad \text{where} \;\; a(s;\nu) := \sqrt{\nu + 2\Gamma(s)},$$

for some $\nu \geq -2\Gamma_{\min}$. The functions $H(s;\nu)$ and $a(s;\nu)$ generalize the functions $H(s)$ and $a(s)$ from section 1.1: $a(s;\mu) = a(s)$ and $H(s;\mu) = H(s)$. The depth $d^* \in (d,\infty]$ is the maximum depth of these trivial flows,

$$d^* = d \cdot \sup_{\nu} H(1;\nu) = d \cdot H(1; -2\Gamma_{\min}).$$

The functions $H(s;\nu)$ play a similar role in our analysis to the linear comparison functions in [11], which considered the irrotational case.

We will need the following lemma concerning the flows $H(s;\nu)$ when proving Proposition 2.1.

LEMMA 2.6. *Define $A\colon (-2\Gamma_{\min},\infty) \to \mathbb{R}$ by*

$$A(\nu) := \begin{cases} \dfrac{1}{2}\,\dfrac{\nu-\mu}{1 - H(1;\nu)} & \nu \neq \mu, \\ \alpha_{\mathrm{cr}} & \nu = \mu. \end{cases}$$

Then A is C^1 and strictly increasing. Moreover, if $d^* = \infty$, then $\lim_{\nu\downarrow -2\Gamma_{\min}} A(\nu) = 0$, and if $d^* < \infty$, then $\lim_{\nu\downarrow -2\Gamma_{\min}} A(\nu) > 0$.

Proof. Differentiating under the integral, we see that $H(1;\nu)$ is a strictly decreasing and strictly convex function of $\nu \in (-2\Gamma_{\min}, \infty)$. Relating $A(\nu)$ to a difference quotient of $H(1;\nu)$, we deduce that A is strictly increasing for $\nu \neq \mu$. Computing

$$\frac{d}{d\nu} H(1;\nu)\Big|_{\nu=\mu} = -\frac{1}{2}\int_0^1 a(s;\mu)^{-3}\, ds = -\frac{1}{2\alpha_{\mathrm{cr}}},$$

we also find that $\lim_{\nu\to\mu} A(\nu) = \alpha_{\mathrm{cr}}$. Finally, if $d^* < +\infty$, then $\mu > -2\Gamma_{\min}$ and $d^* > d$ imply

$$\lim_{\nu\downarrow -2\Gamma_{\min}} A(\nu) = \lim_{\nu\downarrow -2\Gamma_{\min}} \frac{1}{2}\frac{\nu-\mu}{1-H(1;\nu)} = \frac{1}{2}\frac{2\Gamma_{\min}+\mu}{d^*/d-1} > 0,$$

while if $d^* = +\infty$, we obtain $\lim_{\nu\downarrow -2\Gamma_{\min}} A(\nu) = 0$. ◻

In the irrotational case we have $\mu = \alpha_{\mathrm{cr}} = 1$ and $a(s;\nu) = \sqrt{\nu}$, from which one can easily compute $A(\nu) = (\nu + \sqrt{\nu})/2$. Explicit formulas are also available when the vorticity is constant.

2.2. Bounds on the free surface profile. In order to prove Propositions 2.1 and 2.3, we will use the following consequence of the usual maximum principle.

LEMMA 2.7. *Let $\mathscr{D} = \{(x,y) \in \mathbb{R}^2 : 0 < y < f(x)\}$, where f is a continuous function with limits as $x \to \pm\infty$, and suppose that*

$$Lu = a^{ij} D_{ij} u + b^i D_i u$$

is a uniformly elliptic operator with $a^{ij}, b^i \in C^0_{\mathrm{b}}(\overline{\mathscr{D}})$. If $u \in C^2_{\mathrm{b}}(\overline{\mathscr{D}})$ satisfies $u \geq 0$ on $\partial\mathscr{D}$, $Lu \leq 0$ in $\mathscr{D}$, and

$$\limsup_{|x|\to\infty} \sup_{0<y<f(x)} u(x,y) \geq 0,$$

then $u \geq 0$ in $\mathscr{D}$.

Proof. See, for instance, Theorem 19 in [38]. ◻

Proof of Proposition 2.1. Suppose that (ζ, w) is a nontrivial solitary wave with $\zeta \geq 0$. For convenience we work with the variables $\alpha = \alpha_{\mathrm{cr}} - \zeta$ and $h = H(s;\mu) + w$, which satisfy $\alpha \leq \alpha_{\mathrm{cr}}$ and $h \not\equiv H(s;\mu)$.

First we claim that $h(x,1) \geq 1$ for all $x \in \mathbb{R}$. Assume the contrary. Then, since $h(x,1) \to 1$ as $x \to \pm\infty$, $h(x,1)$ must achieve its minimum value at point $x = x_0$. Since $H(1;\nu)$ is a decreasing function with $H(1;\mu) = 1$ and $H(1;\nu) \to 0$ as $\nu \to \infty$, there exists a unique $\nu > \mu$ such that $h(x_0,1) = H(1;\nu) < 1$. Define

$$\varphi(x,s) := h(x,s) - H(s;\nu).$$

A direct computation shows that φ satisfies

$$(1+h_x^2)\varphi_{ss} - 2h_s h_x \varphi_{xs} + h_s^2 \varphi_{xx} + b_1\varphi_x + b_2\varphi_s = 0, \tag{2.2}$$

where

$$b_1 = -\gamma H_s(s;\nu)^3 \varphi_x, \qquad b_2 = 3\gamma H_s(s;\nu)^2 + 3\gamma H_s(s;\nu)\varphi_s + \gamma\varphi_s^2.$$

By assumption, $h_s > 0$ in Ω. Since $h_s \to H_s(s;\mu)$ as $x \to \pm\infty$, and $H_s(s;\mu)$ is uniformly bounded away from 0, we deduce $h_s \geq \delta > 0$ for some $\delta > 0$. Thus (2.2) is a uniformly elliptic equation for φ; indeed, its highest order coefficients satisfy

$$(1+h_x^2)h_s^2 - h_s^2 h_x^2 = h_s^2 \geq \delta^2.$$

By construction, $\varphi \geq 0$ on $s = 1$ and $\varphi = 0$ on $s = 0$. Since $\nu > \mu$, (1.14d) implies

$$\lim_{x\to\pm\infty} \varphi(x,s) = H(s;\mu) - H(s;\nu) \geq 0,$$

uniformly in s. Thus the maximum principle Lemma 2.7 implies $\varphi \geq 0$ in $\overline{\Omega}$.

Since $\varphi(x_0, 1) = 0$ and $\varphi \not\equiv 0$, the Hopf lemma implies

$$\varphi_s(x_0,1) = h_s(x_0,1) - H_s(1;\nu) < 0 \tag{2.3}$$

and hence that $h_s(x_0, 1) < \nu^{-1/2}$. On the other hand, since $h_x(x_0, 1) = 0$ and $h(x_0, 1) = H(1; \nu)$, the nonlinear boundary condition (1.14a) for h at $(x_0, 1)$ gives

$$\frac{1}{2h_s^2(x_0,1)} + \alpha(H(1;\nu) - 1) = \frac{\mu}{2}. \tag{2.4}$$

Combining (2.3) and (2.4) and rearranging we find

$$\alpha > \frac{1}{2}\,\frac{\nu-\mu}{1-H(1;\nu)} = A(\nu),$$

where we have used that $H(1; \nu) - 1 < 0$. Since $\nu > \mu$, Lemma 2.6 implies $\alpha > A(\nu) \geq \alpha_{\mathrm{cr}}$, contradicting our assumption $\alpha \leq \alpha_{\mathrm{cr}}$. Thus $h(x, 1) \geq 1$ for all $x \in \mathbb{R}$.

Since $h(x, 1) \geq 1$, $w = h(x, s) - H(s; \mu)$ satisfies $w \geq 0$ on $s = 1$ and $w = 0$ on $s = 0$. Applying the maximum principle as before, we conclude that $w \geq 0$ in Ω. Thus $w > 0$ in Ω by the strong maximum principle. Now we show that $w(x, 1) > 0$ for all $x \in \mathbb{R}$. Assume for contradiction that $w(x_0, 1) = 0$ for some $x_0 \in \mathbb{R}$. Since $w \geq 0$ in $\overline{\Omega}$ and $w \not\equiv 0$, we can apply the Hopf lemma to obtain

$$w_s(x_0,1) = h_s(x_0,1) - \mu^{-1/2} < 0. \tag{2.5}$$

On the other hand the boundary condition at $(x_0, 1)$ gives $1/2h_s^2(x_0, 1) = \mu/2$, contradicting the strict inequality in (2.5).

We now explain the sense in which the less precise statement in Proposition 2.1 holds. Suppose (u, v, η, F) is a nontrivial solitary wave corresponding to a solution (ζ, w) of (1.19), and that $F \geq 1$. From (1.20) we see that $\zeta \geq 0$. Since (u, v, η, F) is nontrivial, $w \not\equiv 0$, so the above argument implies $w(x, 1) > 0$ for all $x \in \mathbb{R}$, which by the proof of Proposition 1.4 is equivalent to (u, v, η, F) being a wave of elevation. ∎

To prove Proposition 2.2, we use the following theorem from [21].

Theorem 2.8. *Let (ζ, w) solve (1.19a)–(1.19g) with $\zeta > 0$. If $w(x, 1) > 0$ for all $x \in \mathbb{R}$, then $w(x, s)$ is symmetric in x. That is, there exists x_0 such that $w(x, s) = w(2x_0 - x, s)$ for all $(x, s) \in \overline{\Omega}$. Moreover, $w(x, s)$ monotonically decreases on either side of $x = x_0$, $w_x(x, s) < 0$ for $x_0 < x < \infty$ and $0 < s \leq 1$.*

Proof of Proposition 2.2. Since (ζ, w) is a supercritical wave, by Proposition 2.1 it is also a wave of elevation. Thus by Theorem 2.8, w has the desired monotonicity properties. By the proof of Proposition 1.4, this implies symmetry and monotonicity in the (u, v, η) variables. ∎

Proof of Proposition 2.3. For convenience we work with the variables $\alpha = \alpha_{\mathrm{cr}} - \zeta$ and $h = H + w$. Suppose that (ζ, w) is a nontrivial solitary wave with

$$0 < \max w(x,1) \leq M < d^*/d - 1.$$

Then we can find

$$-2\Gamma_{\min} < \nu_M \le \nu < \mu$$

so that $\max h(x,1) = H(1;\nu)$ and $M = H(1;\nu_M)$, where the functions $H(s;\nu)$ were defined in section 2.1. Since h is an increasing function of s and $h(x,1) \to 1$ as $x \to \pm\infty$, we have $\max_\Omega h = h(x_0,1)$ for some point $(x_0,1)$ on the upper boundary. We now apply the maximum principle as in the proof of Proposition 2.1. The function $\varphi(x,s) := h(x,s) - H(s;\nu)$ satisfies the elliptic equation (2.2) to which the maximum principle applies. By construction we have $\varphi \le 0$ on $s = 1$ and $\varphi = 0$ on $s = 0$. Moreover, by (1.14d),

$$\lim_{x\to\pm\infty} \varphi(x,s) = H(s;\mu) - H(s;\nu) < 0.$$

Thus the maximum principle Lemma 2.7 yields $\varphi \le 0$ in Ω. The Hopf lemma then implies $\varphi_s(x_0,1) > 0$ and hence that $h_s(x_0,1) > \nu^{-1/2}$. Plugging this into the boundary condition for h, we find

$$\alpha > \frac{1}{2}\frac{\mu-\nu}{H(1;\nu)-1} = A(\nu) \ge A(\nu_M) =: C > 0, \tag{2.6}$$

where $A(\nu) \ge A(\nu_M) > 0$ by Lemma 2.6. If $d^* < \infty$, then by Lemma 2.6 we can let $M \to d^*/d - 1$ in (2.6) to obtain $\alpha \ge \inf_{(-2\Gamma_{\min},+\infty)} A > 0$. ∎

We remark that in the irrotational case, the first inequality in (2.6) is equivalent to equation 4.4 in [11].

2.3. Lower bound on the pressure. In order to prove Proposition 2.4, we will work with the stream function formulation (1.9) of the water wave problem. In particular, we will apply the maximum principle to the function

$$p = -\frac{|\nabla\psi|^2}{2} - \alpha(y-1) + \Gamma(-\psi) + \frac{\mu}{2},$$

which differs from the physical pressure P by an additive constant and is defined in the fluid domain

$$D_\eta = \{(x,y) \in \mathbb{R}^2 : 0 < y < \eta(x)\}.$$

Because of the boundary condition satisfied by ψ in (1.9a), p vanishes on $y = \eta(x)$.

The following lemma from [46] is an improved version of a similar lemma in [9].

LEMMA 2.9. *Suppose that $\eta \in C^{2+\beta}_{\mathrm{b}}(\mathbb{R})$ and $\psi \in C^{2+\beta}_{\mathrm{b}}(\overline{D_\eta})$ satisfy* (1.9), *$\sup_{D_\eta} \psi_y < 0$, and $\psi_{xx} \to 0$ as $x \to \pm\infty$, uniformly in y. Then p (defined above) satisfies*

$$p \ge -\tfrac{1}{2}|\gamma^+|_0(\psi+1), \tag{2.7}$$

where $\gamma^+ = \max(\gamma,0)$.

Proof. Using $\Delta\psi = -\gamma(\psi)$ we first compute

$$p_x = \psi_x\psi_{yy} - \psi_y\psi_{xy}, \qquad p_y = \psi_y\psi_{xx} - \psi_x\psi_{xy} - \alpha.$$

Combining this with $p = 0$ on $y = \eta(x)$, we have

$$p(x,y) = -\int_y^{\eta(x)} p_y\,dy' = \int_y^{\eta(x)} (\psi_x\psi_{xy} - \psi_y\psi_{xx})\,dy' + \alpha(\eta(x) - y).$$

Thanks to the asymptotic conditions $\psi_x, \psi_{xx} \to 0$ as $x \to \pm\infty$, the integral term vanishes as $x \to \pm\infty$, leaving us with

$$p - \alpha(\eta(x) - y) \to 0 \qquad \text{as } x \to \pm\infty, \tag{2.8}$$

uniformly in y. Taking distributional derivatives, we also find

$$\Delta p = (\Delta\psi)^2 - \psi_{xx}^2 - \psi_{yy}^2 - 2\psi_{xy}^2 \in C_{\mathrm{b}}^{\beta}(\overline{D_\eta})$$

and hence by elliptic regularity that $p \in C_{\mathrm{b}}^{2+\beta}(\overline{D_\eta})$.

A direct computation, depending only on the identity $\Delta\psi = -\gamma(\psi)$, shows that $\varphi = p + \alpha(y-1)$ satisfies

$$\Delta\varphi + \frac{2\varphi_x + 2\gamma\psi_x}{|\nabla\psi|^2}\varphi_x + \frac{2\varphi_y + 2\gamma\psi_y}{|\nabla\psi|^2}\varphi_y = 0.$$

Setting $M = \frac{1}{2}|\gamma^+|_0$, we infer that $\theta = p + M(\psi+1)$ satisfies

$$\Delta\theta + b_1\theta_x + b_2\theta_y = M(\gamma - 2M) - \frac{2\alpha}{|\nabla\psi|^2}\Big((\gamma - 2M)\psi_y + \alpha\Big) \le 0,$$

where

$$b_1 = 2\frac{(\gamma - 2M)\psi_x + \theta_x}{|\nabla\psi|^2}, \qquad b_2 = 2\frac{(\gamma - 2M)\psi_y + \theta_y + 2\alpha}{|\nabla\psi|^2},$$

and where we've used the fact that $\psi_y < 0$. On the free surface $y = \eta(x)$ we have $\theta = M(\psi+1) = 0$. On the bottom $y = 0$ we have

$$\theta_y = \psi_y\psi_{xx} - \psi_x\psi_{xy} - \alpha + M\psi_y = -\alpha + M\psi_y < 0, \tag{2.9}$$

so θ cannot achieve a minimum there. Using (2.8), we also get

$$\lim_{x\to\pm\infty} \inf_{0<y<\eta(x)} \theta = \lim_{x\to\pm\infty} \inf_{0<y<\eta(x)} \Big(\alpha(\eta(x) - y) + M(\psi+1)\Big) \ge 0.$$

Thus the maximum principle Lemma 2.7 yields $\theta \ge 0$ in D_η, from which the proposition follows. $\square$

The inequality (2.9) is one of very few places in this paper where $\alpha > 0$ is important. This inequality corresponds to the gravitational constant g being positive, i.e., to gravity pointing downward.

Proof of Proposition 2.4. Let (ζ, w) be a solitary wave with $\zeta \ge 0$, and set $\alpha = \alpha_{\mathrm{cr}} - \zeta$ for convenience. Since $\inf_\Omega h_s > 0$, the associated stream function ψ will have $\sup \psi_y < 0$. Thanks to the asymptotic condition (1.19f) satisfied by w, we also have $\psi_{xx} \to 0$ as $x \to \pm\infty$, uniformly in y. Thus we can apply Lemma 2.9. In (x,s) variables, the inequality (2.7) becomes

$$p = -\frac{1 + w_x^2}{2(H_s + w_s)^2} - \alpha(H + w - 1) + \frac{1}{2H_s^2} \ge -\frac{1}{2}|\gamma^+|_0(1-s),$$

where we've used the identity $1/H_s^2 = \mu + 2\Gamma(s)$. Rearranging and using the fact that $h = H + w \ge 0$ and $\alpha \le \alpha_{\mathrm{cr}}$, we have

$$\begin{aligned} \frac{1 + w_x^2}{2(H_s + w_s)^2} &\le \frac{1}{2}|\gamma^+|_0(1-s) - \alpha(H + w - 1) + \frac{1}{2H_s^2} \\ &\le \frac{1}{2}|\gamma^+|_0 + \alpha_{\mathrm{cr}} + \frac{1}{2H_s^2} \le C_1, \end{aligned} \tag{2.10}$$

where the constant C_1 depends only on γ. Thus (2.10) implies the pointwise bounds

$$H_s + w_s \geq \frac{1}{\sqrt{2C_1}} =: \delta_*, \qquad |w_x| \leq \sqrt{2C_1}(H_s + w_s) \leq C_2(1 + |w_s|). \qquad \square$$

2.4. Flow force and equidecay. For any solution (α, h) of (1.14a)–(1.14c), the quantity

$$S(\alpha, h) = \int_0^1 \left(\frac{1 - h_x^2}{2h_s^2} - \alpha(h-1) + \Gamma + \frac{\mu}{2} \right) h_s \, ds \tag{2.11}$$

is a constant independent of x. In physical variables, this is, up to rescaling,

$$\int_0^\eta (P + (u-c)^2) \, dy,$$

which is called the *flow force* in [25]. S is also related to the spatial Hamiltonian in [18]. For uniform flows $h = H(s;\nu)$, we compute

$$S(\alpha, H(\,\cdot\,;\nu)) = \frac{\mu - \nu}{2} H(1;\nu) - \frac{\alpha}{2}\big(H(1;\nu) - 1\big)^2 + \frac{\alpha}{2} + \int_0^1 a(s;\nu)\, ds.$$

For solutions h of (1.14), the asymptotic condition (1.14d) implies $S(\alpha, h) = S(\alpha, H)$, where as usual $H(s) = H(s;\mu)$.

We will use the flow force to distinguish $H(s)$ from $H(s;\nu)$ with $\nu \neq \mu$.

LEMMA 2.10. *Suppose h is independent of x and solves* (1.14a)–(1.14c) *with $h_s > 0$ on* $[0,1]$. *Then $S(\alpha, h) > S(\alpha, H)$.*

Proof. Since $h(s)$ solves (2.1) with $h_s > 0$, we must have $h(s) = H(s;\nu)$ for some $\nu \geq -2\Gamma_{\min}$ and hence in particular $S(\alpha, h) = S(\alpha, H(\,\cdot\,;\nu))$. Assuming that $\mu \neq \nu$, we can solve (1.14b) for α to get $\alpha = A(\nu)$, and thus by Lemma 2.6 that $\nu < \mu$. We now compute

$$\frac{\partial}{\partial \tau} S(\alpha, H(\,\cdot\,;\tau)) = H_\tau(1;\tau)(H(1;\tau) - 1)(A(\tau) - \alpha).$$

For $\nu < \tau < \mu$, we have $H(1;\tau) > 1$ and $H_\tau(1;\tau) < 0$, and also by Lemma 2.6 that $A(\tau) > \alpha$. Thus $\frac{\partial}{\partial \tau} S(\alpha, H(\,\cdot\,;\tau)) < 0$ for $\nu < \tau < \mu$, which implies $S(\alpha, h) > S(\alpha, H) = S(\alpha, H(\,\cdot\,;\mu))$. $\square$

Using Lemma 2.10, we can now prove Proposition 2.5. The first step, which is inspired by [39], rephrases equidecay in terms of sequences of translations of waves in $\mathscr{W}$.

Proof of Proposition 2.5. Assume that the proposition is false. Then there exists $(\zeta_n, w_n) \in \mathscr{W}$ and $(x_n, s_n) \in \Omega$ with $x_n \to \infty$ and $|w_n(x_n, s_n)| \geq \varepsilon$ for some fixed $\varepsilon > 0$. Without loss of generality we can assume that $s_n \to s_0 \in [0,1]$ and $\zeta_n \to \zeta \in [0, \alpha_{\mathrm{cr}}]$. For convenience, we work with the variables $h_n = H + w_n$ and $\alpha = \alpha_{\mathrm{cr}} - \zeta$. Consider the translated sequence

$$\hat{h}_n(x, s) = h_n(x + x_n, s).$$

Since $|\hat{w}_n|_{2+\beta}$ and hence $|\hat{h}_n|_{2+\beta}$ are uniformly bounded, we can extract a subsequence so that $\hat{h}_n \to \hat{h}$ in $C^2_{\mathrm{loc}}(\overline{\Omega})$, where $\hat{h} \in C^{2+\beta}_{\mathrm{b}}(\overline{\Omega})$ has $\hat{h} \not\equiv H$. By Proposition 2.4, $\partial_s h_n \geq \delta_*$ for each n, so $\hat{h}_s \geq \delta_*$. Moreover, since $(\alpha_n, \hat{h}_n)$ solves (1.14a)–(1.14c),

$(\alpha, \hat{h})$ solves (1.14a)–(1.14c) as well. Finally, since $\hat{h}$ is obtained as a limit of solitary waves, $S(\alpha, \hat{h}) = S(\alpha, H)$.

Proposition 2.2 implies that the waves in $\mathscr{W}$ are monotone, so $\hat{h}_x \leq 0$. Since $\hat{h}$ is bounded, this forces

$$\hat{h}(x, s) \to H_\pm(s) \qquad \text{as } x \to \pm\infty, \tag{2.12}$$

pointwise in s for some bounded functions $H_\pm$, as well as

$$H_+(s) \leq \hat{h}(x, s) \leq H_-(s) \qquad \text{in } \overline{\Omega}. \tag{2.13}$$

We claim that $H_\pm \in C^{2+\beta}[0, 1]$ solves (1.14a)–(1.14c) with $S(\alpha, H_\pm) = S(\alpha, H)$ and $\partial_s H_\pm \geq \delta_*$. To see this, consider another translated sequence

$$h_n^\star(x, s) = \hat{h}(x + n, s), \qquad n = 1, 2, \ldots.$$

We can extract a subsequence so that $h_n^\star$ converges in $C^2_{\mathrm{loc}}(\overline{\Omega})$ to a function $h^\star$ in $C^{2+\beta}_{\mathrm{b}}(\overline{\Omega})$ solving (1.14a)–(1.14c) with $S(\alpha, h^\star) = S(\alpha, H)$ and $h^\star_s \geq \delta_*$. Then (2.12) implies $h^\star = H_+$, and hence that $H_+ \in C^{2+\beta}[0, 1]$ solves (1.14a)–(1.14c) with $S(\alpha, H_+) = S(\alpha, H)$ and $\partial_s H_+ \geq \delta_*$. The argument for H_- is similar. But by Lemma 2.10, the only function $H_\pm(s)$ satisfying all of these requirements is $H_\pm(s) = H(s)$. Thus (2.13) becomes

$$H(s) \leq \hat{h}(x, s) \leq H(s) \qquad \text{in } \overline{\Omega},$$

which forces $\hat{h} \equiv H$ and hence $\hat{w} \equiv 0$, a contradiction. ∎

3. Properness and spectral properties. In this section, we will formulate (1.19) as a nonlinear operator equation $\mathscr{F}(\zeta, w) = 0$ in weighted Hölder spaces. Our main result is Theorem 3.10, which asserts that $\mathscr{F}$ is locally proper when restricted to $\delta < \zeta < \alpha_{\mathrm{cr}} - \delta$ for any $\delta > 0$. We call a nonlinear mapping $F\colon X \to Y$ *locally proper* if $F^{-1}(K) \cap D$ is compact whenever $K \subset Y$ is compact and $D \subset X$ is closed and bounded. In bounded domains, local properness follows from Schauder estimates, but this argument no longer works in unbounded domains. Many of the lemmas we will need depend only on the domain, ellipticity, and the divergence structure of the equation. While nonstandard, they are fairly straightforward to prove, and we defer them to Appendix A.

In section 3.1, we will introduce the weighted Hölder spaces $C^{k+\beta}_\sigma(\overline{\Omega})$ and define the nonlinear operator $\mathscr{F}$. Here the weight function σ is essentially arbitrary; we only assume symmetry, smoothness, and a subexponential growth condition (3.1). A particular weight function σ will eventually be constructed in sections 5.6–5.7. For $\zeta > 0$, we will show in section 3.2 that the linearized operators $\mathscr{F}_w(\zeta, 0)$ associated with trivial solutions $w \equiv 0$ are invertible and that the general linearized operators $\mathscr{F}_w(\zeta, w)$ are Fredholm with index 0. Since, for linear operators, local properness is equivalent to being semi-Fredholm with index $< +\infty$, i.e. to having a closed range and finite-dimensional kernel, this will also show that the linearized operators $\mathscr{F}_w(\zeta, w)$ are locally proper. In section 3.3, we will define and study the spectra of the linearized operators $\mathscr{F}_w(\zeta, w)$. Finally, in section 3.4 we will prove that $\mathscr{F}$ is locally proper. While the linear arguments in section 3.2 and 3.3 are valid in both weighted and unweighted spaces, this nonlinear argument uses the weight function in a crucial way.

3.1. Formulation in weighted Hölder spaces. In section 3.4 we will need control over the rate at which w, Dw, D^2w decay as $x \to \pm\infty$. For this purpose we introduce the weighted Hölder spaces

$$C^{k+\beta}_\sigma(\overline{\Omega}) = \{u \in C^{k+\beta}(\overline{\Omega}) : |\sigma u|_{k+\beta} < \infty\}.$$

Here the weight function $\sigma \in C^\infty(\mathbb{R})$ is a strictly positive even function satisfying

$$\lim_{x\to\pm\infty} \sigma = \infty, \qquad \lim_{x\to\pm\infty} \frac{D^k_x \sigma}{\sigma} = 0 \quad \text{for } k \geq 1, \tag{3.1}$$

but is otherwise arbitrary. We think of (3.1) as a subexponential growth condition and note that it is satisfied, for instance, by $\sigma(x) = (1+x^2)^{p/2}$ for any $p > 0$.

Let $T = \{s = 1\}$ be the top and $B = \{s = 0\}$ the bottom of the infinite strip $\Omega = \mathbb{R} \times (0,1)$. Introducing the notation

$$C^{k+\beta}_{\sigma,\mathrm{e}}(\overline{\Omega}) = \{u \in C^{k+\beta}_\sigma(\overline{\Omega}) : u \text{ is even in } x\},$$

and similarly for $C^{k+\beta}_{\mathrm{b,e}}(\overline{\Omega})$, we will consider (1.19) as a system for

$$w \in \mathscr{X}^{\mathrm{e}}_\sigma = \{u \in C^{2+\beta}_{\sigma,\mathrm{e}}(\overline{\Omega}) : u = 0 \text{ on } B\}. \tag{3.2}$$

Note that $w \in \mathscr{X}^{\mathrm{e}}_\sigma$ implies the linear conditions (1.19d), (1.19e), (1.19f), and (1.19h).

For supercritical solitary waves (ζ, w), the requirement $w \in \mathscr{X}^{\mathrm{e}}_\sigma$ is not a restriction. This follows from the following result from [21].

THEOREM 3.1. *Let (ζ, w) solve (1.19) with $\zeta > 0$. Then $|\cosh(kx)w|_{2+\beta} < \infty$, where the constant $k > 0$ depends only on ζ.*

LEMMA 3.2. *Let (ζ, w) solve (1.19) with $\zeta > 0$. Then $w \in \mathscr{X}^{\mathrm{e}}_\sigma$.*

Proof. Pick k as in Theorem 3.1. From (3.1) we easily check σ and all of its derivatives grow more slowly than any exponential $Ce^{\varepsilon|x|}$ with $\varepsilon > 0$. Thus $|\sigma/\cosh(kx)|_3 < \infty$, from which the lemma follows. ∎

Next we need to encode (1.19g), $\inf_\Omega(H_s + w_s) > 0$. Setting δ_* as in Proposition 2.4, we know that all solutions (ζ, w) of (1.19) with $\zeta \geq 0$ in fact satisfy the uniform condition $\inf_\Omega(H_s + w_s) \geq \delta_*$. We therefore define

$$\mathscr{U}_\sigma := \left\{ w \in \mathscr{X}^{\mathrm{e}}_\sigma : \inf_\Omega (H_s + w_s) > \frac{\delta_*}{2} \right\}, \tag{3.3}$$

which is an open subset of $\mathscr{X}^{\mathrm{e}}_\sigma$. We now write the remaining nonlinear equations (1.19a)–(1.19b) as $\mathscr{F}(\zeta, w) = 0$, where

$$\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) \colon (0, \alpha_{\mathrm{cr}}) \times \mathscr{U}_\sigma \longrightarrow \mathscr{Y}^{\mathrm{e}}_\sigma$$

is given by

$$\begin{aligned} \mathscr{F}_1(w) &= \left(\frac{w_x}{H_s + w_s} \right)_x + \left(-\frac{1 + w_x^2}{2(H_s + w_s)^2} + \Gamma \right)_s, \\ \mathscr{F}_2(\zeta, w) &= \left(\frac{1 + w_x^2}{2(H_s + w_s)^2} + (\alpha_{\mathrm{cr}} - \zeta) w - \frac{\mu}{2} \right) \bigg|_T, \end{aligned} \tag{3.4}$$

and $\mathscr{Y}^{\mathrm{e}}_\sigma$ is the natural target space of $\mathscr{F}$,

$$\mathscr{Y}^{\mathrm{e}}_\sigma = C^\beta_{\sigma,\mathrm{e}}(\overline{\Omega}) \times C^{1+\beta}_{\sigma,\mathrm{e}}(T). \tag{3.5}$$

By Lemma 3.2, the original system (1.19) for (ζ, w) is equivalent to $\zeta \in (0, \alpha_{\rm cr})$, $w \in \mathscr{U}_\sigma$, and $\mathscr{F}(\zeta, w) = 0$.

As in Appendix A, it will be useful to work in a variety of spaces related to $\mathscr{X}_\sigma^{\rm e}, \mathscr{Y}_\sigma^{\rm e}$ but without decay and without evenness in x. We define

$$\mathscr{X}_{\rm b} = \{u \in C_{\rm b}^{2+\beta}(\overline{\Omega}) : u = 0 \text{ on } B\}, \qquad \mathscr{Y}_{\rm b} = C_{\rm b}^{\beta}(\overline{\Omega}) \times C_{\rm b}^{1+\beta}(T),$$

and let $\mathscr{X}_{\rm b}^{\rm e}$, $\mathscr{Y}_{\rm b}^{\rm e}$ be the corresponding subspaces of functions even in x.

3.2. Local properness of linearized operators. Following Appendices A.3 and A.5, our first step in proving local properness for $\mathscr{F}$ will be to analyze the linear operators

$$\mathscr{L} = (\mathscr{A}, \mathscr{B}) = \mathscr{F}_w(\zeta, w)$$

obtained by taking the Fréchet derivative of $\mathscr{F}$ with respect to w. Fix (ζ, w), and for convenience set $h = H + w$. Letting $D_1 = D_x$ and $D_2 = D_s$, these operators are given in compact divergence form by

$$\mathscr{A}\varphi = D_i(b^{ij} D_j \varphi), \qquad \mathscr{B}\varphi = -b^{2j} D_j \varphi + (\alpha_{\rm cr} - \zeta)\varphi$$

with the usual summation convention, and where the coefficients b^{ij} are

$$\begin{pmatrix} b^{11} & b^{12} \\ b^{21} & b^{22} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{H_s + w_s} & -\dfrac{w_x}{(H_s + w_s)^2} \\ -\dfrac{w_x}{(H_s + w_s)^2} & \dfrac{1 + w_x^2}{(H_s + w_s)^3} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{h_s} & -\dfrac{h_x}{h_s^2} \\ -\dfrac{h_x}{h_s^2} & \dfrac{1 + h_x^2}{h_s^3} \end{pmatrix}.$$

We observe that $b^{ij} \in C_{\rm b}^{1+\beta}(\overline{\Omega})$, provided that $w \in C_{\rm b}^{2+\beta}(\Omega)$ and $\inf_\Omega (H_s + w_s) > 0$. Moreover, $\mathscr{A}$ is uniformly elliptic and $\mathscr{B}$ is uniformly oblique. Indeed,

$$b^{22} = \frac{1 + h_x^2}{h_s^3} \geq |h_s|_0^{-3}, \qquad \det(b^{ij}) = \frac{1 + h_x^2}{h_s^3}\frac{1}{h_s} - \left(\frac{h_x}{h_s^2}\right)^2 = \frac{1}{h_s^4} \geq |h_s|_0^{-4}.$$

If instead of linearizing $\mathscr{F}_2(\zeta, w)$ we had linearized

$$\tilde{\mathscr{F}}_2(\zeta, w) := (H_s + w_s)^2 \mathscr{F}_2(\zeta, w) = (H_s + w_s)^2 \left(\alpha w - \frac{\mu}{2}\right) + \frac{1 + w_x^2}{2},$$

our linearized boundary operator would have been

$$\tilde{\mathscr{B}}\varphi = (2\alpha w - \mu)(H_s + w_s)\varphi_s + w_x \varphi_x + \alpha (H_s + w_s)^2 \varphi.$$

For $\tilde{\mathscr{B}}$ to be uniformly oblique, the condition $\inf_\Omega (H_s + w_s) > 0$ required for uniform ellipticity must be supplemented by $\sup_T (2\alpha w - \mu) < 0$. Being able to drop this extra obliqueness condition $\sup_T (2\alpha w - \mu) < 0$ is an advantage of the divergence formulation introduced in [10]. (In [10], however, this extra condition was unnecessarily imposed.)

While we are primarily interested in $\mathscr{L} = (\mathscr{A}, \mathscr{B})$ as a map $\mathscr{X}_\sigma^{\rm e} \to \mathscr{Y}_\sigma^{\rm e}$, we will also think of it as a map $\mathscr{X}_{\rm b} \to \mathscr{Y}_{\rm b}$ and $\mathscr{X}_{\rm b}^{\rm e} \to \mathscr{Y}_{\rm b}^{\rm e}$. First we give sufficient conditions for $\mathscr{L}$ to be invertible.

LEMMA 3.3. *Fix $\zeta \in \mathbb{R}$ and $w \in \mathscr{X}_{\mathrm{b}}^{\mathrm{e}}$ with $\inf_\Omega(H_s + w_s) > 0$. Suppose that*

$$(\alpha_{\mathrm{cr}} - \zeta)\sup_x \int_0^1 (H_s + w_s)^3\, ds < 1. \tag{3.6}$$

Then the linear operator $\mathscr{L} = \mathscr{F}_w(\zeta, w)$ is invertible $\mathscr{X}_\sigma^{\mathrm{e}} \to \mathscr{Y}_\sigma^{\mathrm{e}}$, $\mathscr{X}_{\mathrm{b}} \to \mathscr{Y}_{\mathrm{b}}$, and $\mathscr{X}_{\mathrm{b}}^{\mathrm{e}} \to \mathscr{Y}_{\mathrm{b}}^{\mathrm{e}}$.

Proof. We observe that $\mathscr{L}\varphi = (f, g)$ has the same divergence form structure as (A.7). Since $(H_s + w_s)^3 = b^{11}/\det(b^{ij})$, condition (3.6) is precisely (A.8), so $\mathscr{L}\colon \mathscr{X}_{\mathrm{b}} \to \mathscr{Y}_{\mathrm{b}}$ is invertible by Lemma A.5. Here we are using the evenness of w in x and Lemma A.13 in order to apply Lemma A.5 in spaces of even functions. We then obtain invertibility $\mathscr{X}_{\mathrm{b}}^{\mathrm{e}} \to \mathscr{Y}_{\mathrm{b}}^{\mathrm{e}}$ by Lemma A.13, and invertibility $\mathscr{X}_\sigma^{\mathrm{e}} \to \mathscr{Y}_\sigma^{\mathrm{e}}$ by Lemma A.11. ∎

COROLLARY 3.4. *For $\zeta > 0$, the linear operator $\mathscr{L} = \mathscr{F}_w(\zeta, 0)$ is invertible $\mathscr{X}_\sigma^{\mathrm{e}} \to \mathscr{Y}_\sigma^{\mathrm{e}}$, $\mathscr{X}_{\mathrm{b}} \to \mathscr{Y}_{\mathrm{b}}$, and $\mathscr{X}_{\mathrm{b}}^{\mathrm{e}} \to \mathscr{Y}_{\mathrm{b}}^{\mathrm{e}}$.*

Proof. With $w = 0$, (3.6) becomes

$$(\alpha_{\mathrm{cr}} - \zeta)\int_0^1 H_s^3\, ds = \frac{\alpha_{\mathrm{cr}} - \zeta}{\alpha_{\mathrm{cr}}} < 1,$$

which holds if and only if $\zeta > 0$, so the statement follows immediately from Lemma 3.3. ∎

COROLLARY 3.5. *For $\zeta > 0$ and $w \in \mathscr{X}_\sigma^{\mathrm{e}}$ with $\inf_\Omega(H_s + w_s) > 0$, the linear operator $\mathscr{L} = \mathscr{F}_w(\zeta, w)$ is Fredholm with index 0 as a map $\mathscr{X}_\sigma^{\mathrm{e}} \to \mathscr{Y}_\sigma^{\mathrm{e}}$, $\mathscr{X}_{\mathrm{b}} \to \mathscr{Y}_{\mathrm{b}}$, and $\mathscr{X}_{\mathrm{b}}^{\mathrm{e}} \to \mathscr{Y}_{\mathrm{b}}^{\mathrm{e}}$.*

Proof. Since $w \in \mathscr{X}_\sigma^{\mathrm{e}}$, the limiting operator (see Appendix A.3) for $\mathscr{L} = \mathscr{F}_w(\zeta, w)$ is $\mathscr{L}_0 = \mathscr{F}_w(\zeta, 0)$. Since $\mathscr{L}_0$ is invertible $\mathscr{X}_{\mathrm{b}} \to \mathscr{Y}_{\mathrm{b}}$ by Corollary 3.4, Lemma A.7 implies that $\mathscr{L}$ is locally proper $\mathscr{X}_{\mathrm{b}} \to \mathscr{Y}_{\mathrm{b}}$, and hence semi-Fredholm with index $\nu < +\infty$. For $t \in [0, 1]$, set

$$\mathscr{L}_t = \mathscr{F}_w(\zeta, tw)\colon \mathscr{X}_{\mathrm{b}} \to \mathscr{Y}_{\mathrm{b}}.$$

Then $\mathscr{L}_t$ depends continuously on t in the operator norm, and, by the above argument, is semi-Fredholm for each t. By the continuity of the index, the index of $\mathscr{L}_t$ is independent of t and hence equal to 0 since $\mathscr{L}_0$ is invertible. In particular, $\mathscr{L} = \mathscr{L}_1$ is Fredholm with index 0 as a map $\mathscr{X}_{\mathrm{b}} \to \mathscr{Y}_{\mathrm{b}}$. $\mathscr{L}$ is then Fredholm with index 0 as a map $\mathscr{X}_{\mathrm{b}}^{\mathrm{e}} \to \mathscr{Y}_{\mathrm{b}}^{\mathrm{e}}$ by Lemma A.13, and Fredholm with index 0 as a map $\mathscr{X}_\sigma^{\mathrm{e}} \to \mathscr{Y}_\sigma^{\mathrm{e}}$ by Lemma A.10. ∎

LEMMA 3.6. *Fix $\zeta > 0$ and $w \in \mathscr{X}_\sigma^{\mathrm{e}}$ satisfying $\inf_\Omega(H_s + w_s) > 0$, and set $(\mathscr{A}, \mathscr{B}) = \mathscr{F}_w(\zeta, w)$. There exists $\kappa_0 < 0$ so that for all $\kappa \in \mathbb{C} \setminus (-\infty, \kappa_0]$ the linear operator $(\mathscr{A} - \kappa I, \mathscr{B})$ is Fredholm with index 0 as a map $\mathscr{X}_\sigma^{\mathrm{e}} \to \mathscr{Y}_\sigma^{\mathrm{e}}$, $\mathscr{X}_{\mathrm{b}} \to \mathscr{Y}_{\mathrm{b}}$, and $\mathscr{X}_{\mathrm{b}}^{\mathrm{e}} \to \mathscr{Y}_{\mathrm{b}}^{\mathrm{e}}$ where we temporarily allow functions in these spaces (but not w itself) to be complex-valued.*

Proof. We argue exactly as in the proofs of Lemma 3.3 and Corollaries 3.4 and 3.5, with Lemma A.6 playing the role of Lemma A.5. ∎

3.3. Spectral properties. In this section we define and analyze the spectrum of $\mathscr{L} = \mathscr{F}_w(\zeta, w)$. For brevity, we only consider $\mathscr{L}$ as a map $\mathscr{X}_\sigma^{\mathrm{e}} \to \mathscr{Y}_\sigma^{\mathrm{e}}$, though it is clear from the proofs that analogous results hold with $\mathscr{X}_\sigma^{\mathrm{e}}$ replaced by $\mathscr{X}_{\mathrm{b}}$ or $\mathscr{X}_{\mathrm{b}}^{\mathrm{e}}$.

LEMMA 3.7. *Let $\delta > 0$ and $K \subset \mathbb{R} \times \mathscr{X}_{\mathrm{b}}^{\mathrm{e}}$ be a closed and bounded set with*

$$\inf_K \zeta > \delta, \qquad \inf_K \inf_\Omega (H_s + w_s) > \delta.$$

Fixing $\theta \in (\pi/2, \pi)$, there exist constants $c_1, c_2 > 0$ such that for all $(\zeta, w) \in K$ and $\kappa \in \mathbb{C}$ with $|\arg\kappa| \le \theta$ and $|\kappa| > c_2$,

$$\begin{aligned} &c_1\big(|\sigma\varphi|_{2+\beta} + |\kappa|^{\beta/2}|\sigma\varphi|_2 + |\kappa|^{1/2}|\sigma\varphi|_{1+\beta} + |\kappa|^{(\beta+1)/2}|\sigma\varphi|_1\big) \\ &\quad\le |\sigma(\mathscr{A} - \kappa I)\varphi|_\beta + |\kappa|^{\beta/2}|\sigma(\mathscr{A} - \kappa I)\varphi|_0 \\ &\qquad + |\sigma\mathscr{B}\varphi|_{1+\beta} + |\kappa|^{\beta/2}|\sigma\mathscr{B}\varphi|_1 + |\kappa|^{1/2}|\sigma\mathscr{B}\varphi|_\beta + |\kappa|^{(\beta+1)/2}|\sigma\mathscr{B}\varphi|_0, \end{aligned}$$

where $(\mathscr{A}, \mathscr{B}) = \mathscr{F}_w(\zeta, w)$.

Proof. Thanks to our weighted Schauder estimate (A.14) we can proceed as in [9]. Introducing a new variable $t \in \mathbb{R}$, we consider the operator

$$\left(\mathscr{A} + \frac{\kappa}{|\kappa|}\partial_t^2, \mathscr{B}\right) : C_\sigma^{2+\beta}(\overline{\Omega} \times \mathbb{R}) \to C_\sigma^\beta(\overline{\Omega} \times \mathbb{R}) \times C_\sigma^{1+\beta}(T \times \mathbb{R}),$$

which, by our choice of κ, is uniformly elliptic and uniformly oblique. Thought of as a function of x and t, σ satisfies condition (A.12) from Appendix A.4. Thus we can apply Lemma A.9 to get a weighted Schauder estimate

$$c|\sigma\widetilde{\varphi}|_{2+\beta} \le \big|\sigma(\mathscr{A} + \tfrac{\kappa}{|\kappa|}\partial_t^2)\widetilde{\varphi}\big|_\beta + |\sigma\mathscr{B}\widetilde{\varphi}|_{1+\beta} \tag{3.7}$$

for $\widetilde{\varphi} \in C_\sigma^{2+\beta}(\overline{\Omega} \times \mathbb{R})$ vanishing on $s = 0$ with c independent of $(\zeta, w) \in K$ and κ with $|\arg\kappa| \le \theta$ and $|\kappa| > c_3$. For $\varphi \in \mathscr{X}_\sigma^{\mathrm{e}}$, we set $\widetilde{\varphi}(x, s, t) = e^{i|\kappa|^{1/2}t}\varphi(x, t)$. Applying (3.7), we obtain

$$c\big|\sigma e^{i|\kappa|^{1/2}t}\varphi\big|_{2+\beta;\overline{\Omega}\times\mathbb{R}} \le \big|\sigma e^{i|\kappa|^{1/2}t}(\mathscr{A} - \kappa I)\varphi\big|_{\beta;\overline{\Omega}\times\mathbb{R}} + \big|\sigma e^{i|\kappa|^{1/2}t}\mathscr{B}\varphi\big|_{1+\beta;T\times\mathbb{R}}.$$

Expanding the definitions of the various norms and using (3.1) yields the desired result for $|\kappa|$ sufficiently large. ∎

Using Lemma 3.7, we can analyze the spectrum of the linear operator $\mathscr{F}_w(\zeta, w)$, which we define as in [19].

DEFINITION 3.8. *Let $L = (A, B)\colon X \to Y_1 \times Y_2$ be a bounded operator between Banach spaces with $X \subset Y_1$. We denote by $\Sigma(A, B)$ the spectrum of A, considered as an unbounded operator $\tilde{A}\colon Y_1 \to Y_1$ with domain $\mathscr{D}(\tilde{A}) = X \cap \ker B$.*

LEMMA 3.9. *Fix $\zeta > 0$ and $w \in \mathscr{X}_\mathrm{b}^\mathrm{e}$ satisfying $\inf_\Omega(H_s + w_s) > 0$, and set $(\mathscr{A}, \mathscr{B}) = \mathscr{F}_w(\zeta, w)\colon \mathscr{X}_\sigma^\mathrm{e} \to \mathscr{Y}_\sigma^\mathrm{e}$. Then there exists an open neighborhood $\mathscr{N}$ of the ray $\{\kappa \in \mathbb{C} : \kappa \ge 0\}$ in $\mathbb{C}$ such that $\Sigma(\mathscr{A}, \mathscr{B}) \cap \mathscr{N}$ consists of finitely many eigenvalues, each with finite algebraic multiplicity.*

Proof. Defining $\tilde{\mathscr{A}}$ as in Definition 3.8, our weighted Schauder estimate (A.14) shows that $\tilde{\mathscr{A}}$ is a closed operator. Pick κ_0 as in Lemma 3.6. Then $\tilde{\mathscr{A}} - \kappa I$ is Fredholm of index 0 whenever $\operatorname{Re}\kappa > \kappa_0$. Letting $\mathscr{N} = \{\kappa : \operatorname{Re}\kappa > \kappa_0\}$ we therefore have by Chapter IV, section 6 of [23] that $\Sigma(\mathscr{A}, \mathscr{B}) \cap \mathscr{N}$ consists of isolated eigenvalues with finite algebraic multiplicities. By Lemma 3.7, $\tilde{\mathscr{A}} - \kappa I$ is one-to-one and hence invertible for κ with $|\arg\kappa| \le 3\pi/4$ and $|\kappa|$ sufficiently large, and so $\Sigma(\mathscr{A}, \mathscr{B}) \cap \mathscr{N}$ is a relatively compact subset of $\mathscr{N}$. In particular, since points in $\Sigma(\mathscr{A}, \mathscr{B}) \cap \mathscr{N}$ are isolated, $\Sigma(\mathscr{A}, \mathscr{B}) \cap \mathscr{N}$ consists of only finitely many eigenvalues. ∎

3.4. Local properness of the nonlinear operator. Finally, we show local properness of the nonlinear operator $\mathscr{F}$ using the results from section 3.2 together with Appendix A.4. Recall the open set $\mathscr{U}_\sigma \subset \mathscr{X}_\sigma^\mathrm{e}$ from section 3.1,

$$\mathscr{U}_\sigma = \left\{w \in \mathscr{X}_\sigma^\mathrm{e} : \inf_\Omega(H_s + w_s) > \frac{\delta_*}{2}\right\},$$

where $\delta_* > 0$ is fixed and given by Proposition 2.4. We note that $\mathscr{U}_\sigma$ and its closure are convex. Picking a small parameter $\delta > 0$, we will deal with solitary waves

$$(\zeta, w) \in (\delta, \alpha_{\mathrm{cr}} - \delta) \times \mathscr{U}_\sigma,$$

so that the usual inequalities $0 < \zeta < \alpha_{\mathrm{cr}}$ hold uniformly.

THEOREM 3.10. *Fix $\delta > 0$. Then $\mathscr{F} \colon [\delta, \alpha_{\mathrm{cr}} - \delta] \times \overline{\mathscr{U}_\sigma} \to \mathscr{Y}_\sigma^{\mathrm{e}}$ is locally proper. Moreover, $w \mapsto \mathscr{F}(\zeta, w)$ is locally proper $\overline{\mathscr{U}_\sigma} \to \mathscr{Y}_\sigma^{\mathrm{e}}$ for any $\zeta > 0$.*

Proof. We simply apply Lemma A.12 and its parameter dependent version. By Lemma A.13, we can ignore the fact that we're dealing with spaces of even functions when applying Lemma A.12.

Fix $\zeta > 0$. We easily check that $\overline{\mathscr{U}_\sigma}$ satisfies the hypotheses on $\mathscr{D}$ in Lemma A.12 and that $w \mapsto \mathscr{F}(\zeta, w)$ has the necessary regularity. By Corollary 3.5, the linear operators $\mathscr{F}_w(\zeta, w)$ for $w \in \mathscr{U}_\sigma$ are Fredholm and hence locally proper. Thus by Lemma A.12, $w \mapsto \mathscr{F}(\zeta, w)$ is locally proper $\mathscr{U}_\sigma \to \mathscr{Y}_\sigma^{\mathrm{e}}$. Similarly, by the parameter-dependent version of Lemma A.12, $\mathscr{F} \colon [\delta, \alpha_{\mathrm{cr}} - \delta] \to \mathscr{Y}_\sigma^{\mathrm{e}}$ is locally proper. ∎

4. Small-amplitude solutions. We now turn our attention to small-amplitude solitary waves. In the notation of section 3.1, these are solutions (ζ, w) of $\mathscr{F}(\zeta, w) = 0$ with $|w|_2$ small. Such waves were constructed in [20] and later in [18]. The object of this section is the following theorem, which asserts the existence and uniqueness of a one-parameter family of small-amplitude solutions (ζ, w^ζ), $0 < \zeta < \zeta_*$, together with the continuous dependence of w^ζ on ζ and the invertibility of the associated linearized operators $\mathscr{F}_w(\zeta, w^\zeta)$. We recall that $\mathscr{F}, \mathscr{X}_\sigma^{\mathrm{e}}, \mathscr{Y}_\sigma^{\mathrm{e}}$ were defined in section 3.1 in terms of a fixed weight function σ satisfying the subexponential growth condition (3.1). Because the natural rates of decay in this section are exponential, this subexponential weight function σ will not play an important role in the analysis.

THEOREM 4.1. *For $\zeta_* > 0$ sufficiently small, there is a one-parameter family (ζ, w^ζ), $0 < \zeta < \zeta_*$ of nontrivial solutions to $\mathscr{F}(\zeta, w) = 0$ with the following properties:*

(i) *(Continuity) The map $\zeta \mapsto w^\zeta$ is continuous from the interval $(0, \zeta_*)$ to $\mathscr{X}_\sigma^{\mathrm{e}}$, and $|w^\zeta|_{2+\beta} \to 0$ as $\zeta \to 0$.*

(ii) *(Invertibility) The linearized operator $\mathscr{F}_w(\zeta, w^\zeta)$ is invertible $\mathscr{X}_{\mathrm{b}}^{\mathrm{e}} \to \mathscr{Y}_{\mathrm{b}}^{\mathrm{e}}$ and $\mathscr{X}_\sigma^{\mathrm{e}} \to \mathscr{Y}_\sigma^{\mathrm{e}}$ for each $0 < \zeta < \zeta_*$.*

(iii) *(Uniqueness) Suppose that (ζ, w) is a nontrivial solution of $\mathscr{F}(\zeta, w)$ with $\zeta > 0$. If ζ and $|w|_2$ are sufficiently small, then $w = w^\zeta$.*

None of the properties (i)–(iii) in Theorem 4.1 are addressed directly in [20] or [18]. Compared with [20], the construction in [18] gives a more detailed description of the small-amplitude solutions, and we will rely heavily on the methods and results of this paper to prove Theorem 4.1. The continuity (i) is relatively straightforward, and we will prove the uniqueness (iii) using the elevation result from section 2. Our main difficulty will be showing the invertibility (ii). Plugging the asymptotic descriptions of w^ζ from [18, 20] into (3.6), it seems that we cannot apply Lemma 3.3, even in the irrotational case. By Corollary 3.5, the linearized operators $\mathscr{F}_w(\zeta, w^\zeta)$ are Fredholm of index 0, so they are invertible if and only if they have trivial kernel. We note that the restriction to spaces of functions even in x here is essential. This is because, for sufficiently smooth solutions of $\mathscr{F}(\zeta, w) = 0$, differentiation with respect to x yields $\mathscr{F}_w(\zeta, w) w_x = 0$.

In section 4.1, we will perform several changes of (dependent) variable which ultimately transform $\mathscr{F}(\zeta, w)$ into an evolution equation $u_x = Lu + N^\zeta(u)$ with x playing the role of time. We will also perform the associated changes of variable in the linearized problems. In section 4.2, we will consider linear equations $v_x = Lv + M(x)v$ with M small. We will show an exponential-dichotomy type result which will allow us to identify solutions with at most mild exponential growth as $x \to \pm\infty$. In section 4.3, we will exhibit the construction of a two-dimensional center manifold controlled by a two-dimensional reduced equation. Homoclinic orbits of this reduced equation correspond to small-amplitude solitary waves. Sections 4.1–4.3 contain results, in particular on linearized problems, which are not present in [18]. We will prove Theorem 4.1 in section 4.4. Using the results on linearized problems from sections 4.1–4.3, we will reduce invertibility (ii) to an elementary fact about the linearizations of two-dimensional equations about homoclinic orbits. To show uniqueness (iii), we will prove that only one homoclinic orbit of the two-dimensional reduced equation can correspond to a wave of elevation and then apply our elevation result, Proposition 2.1.

4.1. Change of variables. In this section we will outline the changes of (dependent) variable in [18] that transform the nonlinear operator equation $\mathscr{F}(\zeta, w) = 0$ into an evolution equation. In addition, we will describe how these changes of variable affect the linearized problems $\mathscr{F}_w(\zeta, w)\varphi = 0$. The explicit changes of variable will be given in the subsections 4.1.1 and 4.1.2.

It is convenient to first change variables in the problem $\mathscr{F}_w(0,0)\varphi = 0$ obtained by linearizing about the critical trivial solution $(\zeta, w) = (0, 0)$,

$$(a^3\varphi_s)_s + (a\varphi_x)_x = 0 \quad \text{in } \Omega, \qquad -a^3\varphi_s + \alpha_{\mathrm{cr}}\varphi = 0 \quad \text{on } T, \qquad \varphi = 0 \quad \text{on } B, \tag{4.1}$$

where $a(s) = 1/H_s$ was defined in (1.24). In this section, we will work with the Hilbert spaces

$$\begin{aligned} X &= \{(w, \theta) \in H^1(0,1) \times L^2(0,1) : w(0) = 0\}, \\ Y &= \{(w, \theta) \in H^2(0,1) \times H^1(0,1) : w(0) = 0\}, \end{aligned}$$

which we think of as spaces of functions of the vertical variable $s \in [0, 1]$. Setting $\vartheta = a\varphi_x$ and thinking of (φ, ϑ) as a mapping $\mathbb{R} \to Y$, (4.1) becomes the linear evolution equation

$$(\varphi, \vartheta)_x = L(\varphi, \vartheta), \tag{4.2}$$

where

$$L\colon \mathscr{D}(L) \subset X \to X, \qquad L\begin{pmatrix}\varphi \\ \vartheta\end{pmatrix} = \begin{pmatrix} a^{-1}\vartheta \\ -(a^3\varphi_s)_s \end{pmatrix}, \tag{4.3}$$

is a closed operator whose domain

$$\mathscr{D}(L) = \big\{(\varphi, \vartheta) \in Y : \vartheta(0) = 0,\ -a^3\varphi_s(1) + \alpha_{\mathrm{cr}}\varphi(1) = 0\big\} \tag{4.4}$$

captures the boundary condition on $s = 1$. We give $\mathscr{D}(L)$ the graph norm, which is equivalent to the Y norm. Evenness of φ in x is now expressed as

$$(\varphi, \vartheta)(-x) = S(\varphi, \vartheta)(x) := (\varphi, -\vartheta)(x), \tag{4.5}$$

where $S(\varphi, \vartheta) = (\varphi, -\vartheta)$ is called the *reverser*. Solutions (φ, ϑ) of (4.2) with this symmetry are called *reversible*.

We now turn to the full nonlinear problem $\mathscr{F}(\zeta, w) = 0$, which we will transform into a nonlinear perturbation of (4.2), written

$$u_x - Lu = N^\zeta(u), \tag{4.6}$$

for $u\colon \mathbb{R} \to X$, provided $\|u\|_Y$ and $|\zeta|$ are sufficiently small. We note that the u appearing in (4.6) is not the horizontal component u of the velocity field discussed in section 1. The Hamiltonian structure of (4.6) plays an important role in [18] but is not needed in our analysis.

To state the main results of this transformation, we introduce the notation

$$\begin{aligned}
C^k_{\mathrm{b}}(\mathbb{R}, E) &= \{u \in C^k(\mathbb{R}, E) : \|u\|_{C^k(\mathbb{R},E)} < \infty\},\\
C^k_0(\mathbb{R}, E) &= \{u \in C^k(\mathbb{R}, E) : D^\ell_x u(x) \to 0 \text{ as } x \to \pm\infty \ \forall \ell \le k\},\\
C^k_\sigma(\mathbb{R}, E) &= \{u \in C^k(\mathbb{R}, E) : \|\sigma u\|_{C^k(\mathbb{R},E)} < \infty\}
\end{aligned}$$

for open subsets E of Banach spaces. Thanks to the subexponential growth condition (3.1) on σ, there exist constants $C_1, C_2 > 0$ depending only on σ so that

$$C_1 \sum_{\ell=1}^{k} \|\sigma D^\ell_x u\|_{C^0(\mathbb{R},E)} \le \|\sigma u\|_{C^k(\mathbb{R},E)} \le C_2 \sum_{\ell=1}^{k} \|\sigma D^\ell_x u\|_{C^0(\mathbb{R},E)}$$

for all $u \in C^k_\sigma(\mathbb{R}, E)$.

Our first lemma asserts that solutions (ζ, w) of $\mathscr{F}(\zeta, w) = 0$ with $|\zeta|$ and $|w|_2$ sufficiently small yield solutions u of $u_x - Lu = N^\zeta(u)$.

LEMMA 4.2. *There exist neighborhoods $\Lambda \subset \mathbb{R}$, $V \subset Y$, and $U \subset \mathscr{D}(L)$ of the origin and smooth maps*

$$G^\zeta\colon V \to Y, \qquad N^\zeta\colon U \to X,$$

defined for $\zeta \in \Lambda$ with the following properties:

(i) *N^ζ is nonlinear in that $N^\zeta(0) = 0$ and $DN^0(0) = 0$.*

(ii) *G^ζ is a near identity transformation in that $G^\zeta(0) = 0$ and $DG^0(0) = I$.*

(iii) *N^ζ and G^ζ respect reversibility in that $N^\zeta \circ S = -S \circ N^\zeta$ and $G^\zeta \circ S = S \circ G^\zeta$.*

(iv) *Let (ζ, w) be a solution of $\mathscr{F}(\zeta, w) = 0$ with $|\zeta|$ and $|w|_2$ sufficiently small, and set*

$$u = G^\zeta\left(w, \frac{w_x}{H_s + w_s}\right). \tag{4.7}$$

Then $u \in C^1_\sigma(\mathbb{R}, X) \cap C^0_\sigma(\mathbb{R}, U)$ solves $u_x - Lu = N^\zeta(u)$ and $u(-x) = Su(x)$.

The second lemma asserts that reversible solutions u of $u_x - Lu = N^\zeta(u)$ with sufficient regularity and which decay as $x \to \pm\infty$ correspond to solutions w of $\mathscr{F}(\zeta, w) = 0$.

LEMMA 4.3. *Suppose that $u \in C^3_\sigma(\mathbb{R}, X) \cap C^2_\sigma(\mathbb{R}, U)$ solves $u_x - Lu = N^\zeta(u)$ and $u(-x) = Su(x)$. Then u is given by (4.7), where $w \in \mathscr{X}^{\mathrm{e}}_\sigma$ solves $\mathscr{F}(\zeta, w) = 0$. The correspondence $u \mapsto w$ is continuous $C^3_\sigma(\mathbb{R}, X) \cap C^2_\sigma(\mathbb{R}, V) \to C^{2+\beta}_\sigma(\overline{\Omega})$ and $C^3_{\mathrm{b}}(\mathbb{R}, X) \cap C^2_{\mathrm{b}}(\mathbb{R}, V) \to C^{2+\beta}_{\mathrm{b}}(\overline{\Omega})$, depending continuously on ζ in both cases.*

The last lemma relates the corresponding linearized problems.

LEMMA 4.4. *In the setting of Lemma 4.3, suppose that $\varphi \in \mathscr{X}^{\mathrm{e}}_{\mathrm{b}}$ is a nontrivial solution of the linearized problem $\mathscr{F}_w(\zeta, w)\varphi = 0$. Then there is a corresponding nontrivial solution v in $C^1_{\mathrm{b}}(\mathbb{R}, X) \cap C^0_{\mathrm{b}}(\mathbb{R}, \mathscr{D}(L))$ of the linearized problem $v_x - Lv = DN^\zeta(u)v$ with $v(-x) = Sv(x)$.*

The rest of section 4.1 is devoted to proving Lemmas 4.2–4.4. In section 4.1.1, we will perform a simple change of variables $w \mapsto (w,\theta)$ which allows us to interpret $\mathscr{F}(\zeta, w) = 0$ as an evolution equation with a nonlinear constraint. In section 4.1.2, we will perform a more complicated change of variables $u = G^\zeta(w,\theta)$, which transforms the previous evolution equation into one with only linear constraints, and prove Lemmas 4.2–4.4. Our arguments are relatively straightforward and follow [18] very closely, but there are several technical details which need to be checked. The reader uninterested these technicalities is encouraged to skip sections 4.1.1–4.1.2 and move on to section 4.2.

4.1.1. First change of variables. In this section we will perform the simple change of variables

$$w \mapsto (w,\theta), \qquad \theta = \frac{w_x}{H_s + w_s}.$$

Recall the basic equations (1.19a)–(1.19b):

$$\tag{4.8} \begin{cases} \left(\dfrac{w_x}{H_s+w_s}\right)_x + \left(-\dfrac{1+w_x^2}{2(H_s+w_s)^2} + \Gamma\right)_s = 0 & \text{on } \Omega, \\ \dfrac{1+w_x^2}{2(H_s+w_s)^2} + (\alpha_{\mathrm{cr}} - \zeta)w - \dfrac{\mu}{2} = 0 & \text{on } T, \end{cases}$$

which we regard in this section as a system for $\zeta \in \mathbb{R}$ and $w \in \mathscr{X}_{\mathrm{b}}^{\mathrm{e}}$ satisfying $\inf_\Omega(H_s + w_s) > 0$. In terms of (w,θ), this can be rewritten as

$$\tag{4.9} \begin{cases} w_x = (H_s + w_s)\theta & \text{in } \Omega, \\ \theta_x = \frac{1}{2}\left(\theta^2 + (H_s+w_s)^{-2}\right)_s - \gamma & \text{in } \Omega, \\ \frac{1}{2}\left(\theta^2 + (H_s+w_s)^{-2}\right) + (\alpha_{\mathrm{cr}} - \zeta)w - \frac{\mu}{2} = 0 & \text{on } s = 1 \end{cases}$$

with $\theta \in C_{\mathrm{b}}^{1+\beta}(\overline{\Omega})$ vanishing on B and odd in x.

The advantage of (4.9) is that it can be interpreted as an autonomous evolution equation with x playing the role of time. To make this explicit, let $V \subset Y$ be a bounded neighborhood of the origin in Y, small enough that $H_s + w_s > \delta_*$ whenever $(w,\theta) \in V$, where $\delta_* > 0$ is given in Proposition 2.4. We then think of (4.9) as the autonomous evolution equation

$$\tag{4.10} (w,\theta)_x = \mathscr{K}_1(w,\theta), \qquad \mathscr{K}_2^\zeta(w,\theta) = 0, \qquad \mathscr{K}_3(w,\theta) = 0,$$

where $\mathscr{K}_1 \colon V \to X$ describes the first two lines of (4.9),

$$\mathscr{K}_1 \begin{pmatrix} w \\ \theta \end{pmatrix} = \begin{pmatrix} (H_s+w_s)\theta \\ \frac{1}{2}(\theta^2 + (H_s+w_s)^{-2})_s - \gamma \end{pmatrix},$$

$\mathscr{K}_2^\zeta \colon V \to \mathbb{R}$ describes the nonlinear boundary condition on $s = 1$,

$$\mathscr{K}_2^\zeta \begin{pmatrix} w \\ \theta \end{pmatrix} = \left(\frac{1}{2}(\theta^2 + (H_s+w_s)^{-2}) + (\alpha_{\mathrm{cr}} - \zeta)w - \frac{\mu}{2} \right)\bigg|_{s=1},$$

and $\mathscr{K}_3 \colon V \to \mathbb{R}$, $\mathscr{K}_3(w,\theta) = \theta(0)$ encodes the remaining boundary condition not dealt with by the definitions of X and Y. We easily check that the maps $\mathscr{K}_1, \mathscr{K}_2^\zeta, \mathscr{K}_3$

are smooth. The evenness of w in x is now expressed as the reversibility $(w,\theta)(-x) = S(w,\theta)(x)$, where S is the reverser from (4.5).

LEMMA 4.5. *Suppose that $\mathscr{F}(\zeta, w) = 0$ and that $|w|_2$ is sufficiently small, and set $\theta = w_x/(H_s + w_s)$. Then $(w,\theta) \in C^1_0(\mathbb{R}, X) \cap C^0_0(\mathbb{R}, V)$ is a reversible solution of (4.10) with $\|(w,\theta)(x)\|_Y \le C|w|_2$.*

Proof. The regularity $(w,\theta) \in C^1(\mathbb{R}, X) \cap C^0(\mathbb{R}, Y)$ and reversibility are straightforward. By Proposition 2.4, we know that $\inf_\Omega(H_s + w_s) \ge \delta_* > 0$, where δ_* is independent of (ζ, w). From this it is easy to show

$$\sup_x \|(w,\theta)(x)\|_Y \le C|w|_2(1 + |w|_2).$$

Thus, if $|w|_2$ is sufficiently small, $(w,\theta)(x)$ will lie in V for all x. In particular, (w,θ) solves (4.10). Since $w, Dw, D^2w \to 0$ uniformly in s as $x \to \pm\infty$, we check that $\|(w,\theta)\|_Y \to 0$ as $x \to \pm\infty$. Since $(w,\theta)_x = \mathscr{K}_1(w,\theta)$ and $\mathscr{K}_1(0,0) = 0$, we conclude that $\|(w,\theta)_x\|_X \to 0$ as $x \to \pm\infty$ as well. □

The following technical lemma asserts that solutions $(w,\theta) \in C^3_\sigma(\mathbb{R}, X) \cap C^2_\sigma(\mathbb{R}, Y)$ of (4.10) give classical solutions $w \in C^{2+\beta}(\overline{\Omega})$ of (4.8).

LEMMA 4.6. *Suppose that $(w,\theta) \in C^3_\sigma(\mathbb{R}, X) \cap C^2_\sigma(\mathbb{R}, V)$ is a reversible solution of (4.10). Then $w \in \mathscr{X}^{\mathrm{e}}_\sigma$ solves $\mathscr{F}(\zeta, w) = 0$. Moreover this correspondence $(w,\theta) \mapsto w$ is continuous both from $C^3_\sigma(\mathbb{R}, X) \cap C^2_\sigma(\mathbb{R}, V)$ to $C^{2+\beta}_\sigma(\overline{\Omega})$ and from $C^3_{\mathrm{b}}(\mathbb{R}, X) \cap C^2_{\mathrm{b}}(\mathbb{R}, V)$ to $C^{2+\beta}_{\mathrm{b}}(\overline{\Omega})$.*

Proof. Since $\beta \in (0, 1/2]$, the Sobolev embeddings $H^2(0,1) \to C^{1+\beta}(0,1)$ and $H^1(0,1) \to C^\beta(0,1)$ immediately imply

$$|(\sigma w)_x|_{1+\beta;\Omega} + |\sigma w|_{1+\beta;\Omega} \le C(\|\sigma(w,\theta)\|_{C^3(\mathbb{R},X)} + \|\sigma(w,\theta)\|_{C^2(\mathbb{R},V)})$$

for any (w,θ). This is the only place where the assumption $\beta \in (0, 1/2]$ in Theorem 1.1 is used. It remains to estimate $|\sigma w_{ss}|_\beta$, and this is where we use the equation, (4.9), satisfied by (w,θ). Setting $h = H + w$ for convenience, we use (4.9) to eliminate θ, finding

$$w_{ss} = \frac{-h_s^2 w_{xx} + 2h_s w_x w_{xs} + \gamma H_s^3 w_x^2 - 3\gamma H_s^2 w_s - 3\gamma H_s w_s^2 - \gamma w_s^3}{1 + w_x^2}. \tag{4.11}$$

Multiplying (4.11) by σ, we obtain an estimate of the form

$$|\sigma w_{ss}|_\beta \le C(1 + |\sigma w|_{1+\beta} + |\sigma w_x|_{1+\beta})^3,$$

where we've used the multiplicative inequality $|f|_{k+\beta} \le |\sigma^{-1}|_{k+\beta}|\sigma f|_{k+\beta}$. Since (w,θ) is reversible, w is even in x, so we have $w \in \mathscr{X}^{\mathrm{e}}_\sigma$. Eliminating θ from (4.9) we then obtain $\mathscr{F}(\zeta, w) = 0$ as desired.

To show continuity in weighted spaces, suppose that (w^i, θ^i) solve (4.10) with $\zeta = \zeta^i$ for $i = 1, 2$. Setting $\varphi = w^1 - w^2$, we can estimate $|\sigma\varphi|_{1+\beta}$ and $|\sigma\varphi_x|_{1+\beta}$ as before, so it remains to estimate $|\sigma\varphi_{ss}|_\beta$. Using the equation to solve for w^i_{ss} as before, we subtract the two expressions and obtain

$$|\sigma\varphi_{ss}|_\beta \le C(1 + |w^1|_{2+\beta} + |w^2|_{2+\beta})^4(|\sigma\varphi|_{1+\beta} + |\sigma\varphi_x|_{1+\beta}).$$

Setting $\sigma = 1$, the same argument gives continuity in unweighted spaces. □

For later reference we also linearize (4.8) with respect to w. Setting $h = H + w$, this yields the system

$$\begin{cases} \partial_s\Big(\dfrac{1+w_x^2}{h_s^3}\varphi_s - \dfrac{w_x}{h_s^2}\varphi_x\Big) + \partial_x\Big(-\dfrac{w_x}{h_s^2}\varphi_s + \dfrac{1}{h_s}\varphi_x\Big) = 0 & \text{in } \Omega, \\ \dfrac{1+w_x^2}{h_s^3}\varphi_s + \dfrac{w_x}{h_s}\varphi_x + (\alpha_{\mathrm{cr}} - \zeta)\varphi = 0 & \text{on } s = 1 \end{cases} \tag{4.12}$$

for $\varphi \in \mathscr{X}_{\mathrm{b}}^{\mathrm{e}}$. Suppose that φ solves (4.12), and set

$$\vartheta = \frac{\varphi_x}{h_s} - \theta\frac{\varphi_s}{h_s}.$$

Then (φ, ϑ) solves the system obtained by linearizing (4.9) about (w, θ),

$$\begin{cases} \varphi_x = h_s\vartheta + \theta\varphi_s & \text{in } \Omega, \\ z_x = \big(wz - h_s^{-3}\varphi_s\big)_s & \text{in } \Omega, \\ \big(wz - h_s^{-3}\varphi_s\big) + (\alpha_{\mathrm{cr}} - \zeta)\varphi = 0 & \text{on } s = 1 \end{cases} \tag{4.13}$$

with $\vartheta \in C_{\mathrm{b}}^{1+\beta}(\overline{\Omega})$ vanishing on B and odd in x. If $(w, \theta) \in V$ for all x, we easily check that the linearized problem (4.13) can be written

$$(\varphi, \vartheta)_x = D\mathscr{K}_1(w,\theta)(\varphi,\vartheta), \qquad D\mathscr{K}_2^\zeta(w,\theta)(\varphi,\vartheta) = 0, \qquad D\mathscr{K}_3(w,\theta)(\varphi,\vartheta) = 0, \tag{4.14}$$

where the Fréchet derivatives of $\mathscr{K}_1, \mathscr{K}_2^\zeta, \mathscr{K}_3$ are taken in Y.

4.1.2. Second change of variables. With (w, θ) as in the previous section, we now make another change of variable $u = G^\zeta(w, \theta)$. Given $(w, \theta) \in Y$, define

$$\begin{aligned} \Xi &= w + a^{-3}(1)s \int_s^1 \left[\frac{1}{2}\left(\theta^2 + \frac{1}{(a^{-1} + w_s)^2}\right) + a^3 w_s - \frac{a^2}{2}\right] ds', \\ \xi &= \Xi - \zeta a^{-3}(1)s \int_s^1 \Xi\, ds'. \end{aligned}$$

The following lemma shows that

$$G^\zeta \colon V \to Y, \qquad G^\zeta(w,\theta) = (\xi, \theta) = u$$

is a valid change of variables for $|\zeta|$ and $V \subset Y$ sufficiently small. Its proof relies on the easily verifiable properties

$$G^\zeta(0,0) = 0, \qquad DG^0(0,0) = \mathrm{id} \colon Y \to Y. \tag{4.15}$$

Lemma 4.7. *For a sufficiently small neighborhood $\Lambda \times V$ of the origin in $\mathbb{R} \times Y$, the following holds:*

(i) *For each $\zeta \in \Lambda$, $G^\zeta \colon V \to Y$ is a diffeomorphism onto its image. The mappings G^ζ and $(G^\zeta)^{-1}$ depend smoothly on $\zeta \in \Lambda$.*

(ii) *For each $(\zeta, w, \theta) \in \Lambda \times V$, the derivative $DG^\zeta(w,\theta) \colon Y \to Y$ extends to an isomorphism $\widehat{DG^\zeta}(w,\theta) \colon X \to X$. The operators $\widehat{DG^\zeta}(w,\theta)$ and $\widehat{DG^\zeta}(w,\theta)^{-1}$ depend smoothly on (ζ, w, θ) in $\Lambda \times V$.*

Proof. This is Lemma 3.2 in [18], so we only give a sketch. Since G^ζ is smooth, the first part of the lemma follows from $DG^0(0) = I$ and $G^\zeta(0) = 0$ by the implicit function theorem. The extension $\widehat{DG}^\zeta(w,\theta)\colon X \to X$ is straightforward, as is the smoothness of $\widehat{DG}^\zeta$ in (ζ, w, θ). The properties of $\widehat{DG}^\zeta(w,\theta)^{-1}$ then follow from applying the implicit function theorem to the equations $\widehat{DG}^\zeta T = I$ and $T\widehat{DG}^\zeta = I$ for $T \in \mathscr{L}(X,X)$, the space of bounded linear operators $X \to X$. ◻

A consequence of Lemma 4.7 and (4.15) is the following technical lemma.

LEMMA 4.8. *For any integer $k \geq 0$, the map $(G^\zeta_*(w,\theta))(x) = G^\zeta(w(x),\theta(x))$ is a homeomorphism both*

$$\begin{aligned} &C^{k+1}_{\mathrm{b}}(\mathbb{R},X) \cap C^k_{\mathrm{b}}(\mathbb{R},V) \to C^{k+1}_{\mathrm{b}}(\mathbb{R},X) \cap C^k_{\mathrm{b}}(\mathbb{R},G^\zeta(V)) \\ \text{and}\ \ &C^{k+1}_{\sigma}(\mathbb{R},X) \cap C^k_{\sigma}(\mathbb{R},V) \to C^{k+1}_{\sigma}(\mathbb{R},X) \cap C^k_{\sigma}(\mathbb{R},G^\zeta(V)), \end{aligned}$$

in each case also depending continuously on $\zeta \in \Lambda$.

Proof. Fix $k \geq 0$, and let $v = (w,\theta) \in C^{k+1}(\mathbb{R},X) \cap C^k(\mathbb{R},V)$. The statement follows from writing

$$\begin{aligned} D^\ell_x G^\zeta(v) &= DG^\zeta(v) D^\ell_x v + R^\zeta_\ell(v, D_x v, \dots, D^{\ell-1}_x v), \qquad 0 \leq \ell \leq k, \\ D^{k+1}_x G^\zeta(v) &= \widehat{DG}^\zeta(v) D^{k+1}_x v + R^\zeta_{k+1}(v, D_x v, \dots, D^k_x v), \end{aligned}$$

and observing that the remainder terms R^ζ_ℓ are smooth $\Lambda \times V \times Y^{\ell-1} \to Y$ and satisfy $R^\zeta_\ell(0,\dots,0) = 0$ and $DR^\zeta_\ell(0,\dots,0) = 0$. ◻

We now plug $(w,\theta) = (G^\zeta)^{-1}(u)$ into (4.10) and obtain a system for u. A direct computation shows

$$\mathscr{K}_2^\zeta(w,\theta) = -a^3\xi_s(1) + \alpha_{\mathrm{cr}}\xi(1), \qquad L = D\mathscr{K}_1(0,0)|_{\mathscr{D}(L)}. \tag{4.16}$$

In particular, in the $u = (\xi,\theta)$ variables, the boundary condition $\mathscr{K}_2^\zeta(w,\theta) = 0$ is both linear and independent of ζ. Defining

$$N^\zeta(u) := D\mathscr{K}_1(0,0)u - DG^\zeta(w,\theta)\mathscr{K}_1(w,\theta), \tag{4.17}$$

(4.16) implies that (4.10) is equivalent to $u_x - Lu = N^\zeta(u)$, where $\mathscr{K}_2^\zeta(w,\theta) = 0$ and $\mathscr{K}_3(w,\theta) = 0$ are captured by requiring $u \in \mathscr{D}(L)$. From (4.17) we see that N^ζ is smooth jointly in u and ζ, defined on a neighborhood of the origin in Y with values in X. Similar computations show that the linearized problem (4.14) for (φ,ϑ) is equivalent to $v_x - Lv = DN^\zeta(u)v$, where v is given by $v = DG^\zeta(w,\theta)(\varphi,\vartheta)$.

The proofs of Lemmas 4.2–4.4 are now straightforward.

Proof of Lemma 4.2. The smoothness of G^ζ and $(G^\zeta)^{-1}$ was shown in Lemma 4.7. The regularity of N^ζ then follows from its definition (4.17) and the smoothness of $\mathscr{K}_1$. We've also already seen $G^\zeta(0) = 0$ and $DG^0(0) = I$; it was (4.15). Combining this with (4.17), we get $N^\zeta(0) = 0$ and $DN^0(0) = 0$. The symmetry $G^\zeta \circ S = S \circ G^\zeta$ follows directly from the definition of G^ζ given at the start of section 4.1.2. From the definition of $\mathscr{K}$ we also have $\mathscr{K}_1 \circ S = -S \circ \mathscr{K}_1$, $\mathscr{K}_2^\zeta \circ S = \mathscr{K}_2^\zeta$, and $\mathscr{K}_3 \circ S = \mathscr{K}_3$, which, combined with (4.17), yields $N^\zeta \circ S = -S \circ N^\zeta$.

Now suppose that $\mathscr{F}(\zeta,w) = 0$, and set $\theta = w_x/(H_s + w_s)$. By Lemma 4.5, if $|w|_2$ is sufficiently small, $(w,\theta) \in C^1_0(\mathbb{R},X) \cap C^0_0(\mathbb{R},V)$ solves the evolution equation (4.10). Since w is even, we also have the reversibility $(w,\theta)(-x) = S(w,\theta)(x)$. Assuming $|\zeta|$ is also sufficiently small, we can define $u = G^\zeta(w,\theta)$, which solves $u_x - Lu = N^\zeta(u)$.

Using $G^\zeta \circ S = S \circ G^\zeta$ we check that u is also reversible, $u(-x) = Su(x)$. Finally, since $G^\zeta(0,0) = 0$, we have $u \in C_0^1(\mathbb{R}, X) \cap C_0^0(\mathbb{R}, V)$. ∎

Proof of Lemma 4.3. Suppose $u \in C_\sigma^3(\mathbb{R}, X) \cap C_\sigma^2(\mathbb{R}, U)$ is a reversible solution of $u_x - Lu = N^\zeta(u)$. Combining Lemma 4.8 and Lemma 4.6 we see that $u = G^\zeta(w, \theta)$, where $w \in \mathscr{X}_\sigma^{\mathrm{e}}$ solves (4.8), and that the correspondence $u \mapsto w$ has the desired continuity properties. ∎

Proof of Lemma 4.4. Let u, w be as in Lemma 4.3, and suppose that $\varphi \in \mathscr{X}_\sigma^{\mathrm{e}}$ is a nontrivial solution of the linearized problem $\mathscr{F}_w(\zeta, w)\varphi = 0$. Set

$$h = H + w, \qquad \theta = \frac{w_x}{h_s}, \qquad \vartheta = \frac{\varphi_x}{h_s} - \theta\frac{\varphi_s}{h_s}, \qquad v = DG^\zeta(w,\theta)(\varphi, \vartheta).$$

Since $DG^\zeta(w, \theta)$ is invertible and $\varphi \not\equiv 0$, we have $v \not\equiv 0$. We know from section 4.1.1 that $(\varphi, \vartheta) \in C_\sigma^1(\mathbb{R}, X) \cap C_\sigma^0(\mathbb{R}, \mathscr{D}(L))$ solves the linearized problem (4.14), so by section 4.1.2 we have $v_x - Lv = DN^\zeta(u)v$. The definitions of θ and ϑ together with $w, \theta \in C_\sigma^{2+\beta}(\overline{\Omega})$ imply that $\|(w,\theta)\|_Y$ and $\|(\varphi, \vartheta)\|_Y$ are bounded uniformly in x, so, by the smoothness of G^ζ, $\|v(x)\|_Y$ is also bounded uniformly in x. Using the equation solved by v, the smoothness of N^ζ, and the boundedness of u, we then have

$$\|v_x\|_X = \|DN^\zeta(u)v + Lv\|_X \le C(\|v\|_X + \|v\|_Y)$$

uniformly bounded in x, so $v \in C_{\mathrm{b}}^1(\mathbb{R}, X) \cap C_{\mathrm{b}}^0(\mathbb{R}, \mathscr{D}(L))$. Finally, from the evenness of φ in x and $G^\zeta \circ S = S \circ G^\zeta$, we conclude that v is reversible, $v(-x) = Sv(x)$. ∎

4.2. Linearization about trivial solutions. In this section we will consider the linear operator L defined by (4.3)–(4.4) in more detail, as well as inhomogeneous linear systems $v_x - Lv = g$ and nonautonomous systems $v_x - Lv = M(x)v$.

The operator L is strongly related to the Sturm–Liouville problem

$$-(a^3\eta')' = \nu a \eta, \qquad -a^3\eta'(1) + \alpha_{\mathrm{cr}}\eta(1) = 0, \qquad \eta(0) = 0, \tag{4.18}$$

for $\eta \in C^2[0,1]$ and $\nu \in \mathbb{R}$. We begin with a lemma from [18, Lemma 3.3 and following discussion] on the spectrum of L.

LEMMA 4.9. *Let $\nu_0 < \nu_1 < \cdots$ be the eigenvalues of* (4.18)*. Then $\nu_0 = 0$, and the following hold:*

(i) *The spectrum of $L\colon \mathscr{D}(L) \subset X \to X$ consists of the algebraically simple eigenvalues $\{\pm\sqrt{\nu_k}\}_{k=1}^\infty$ together with an eigenvalue at* 0 *with algebraic multiplicity* 2*. The generalized eigenvectors u_1, u_2 with $Lu_1 = 0$ and $Lu_2 = u_1$ are*

$$u_1 = \left(\int_0^s a^{-3}(t)\,dt,\ 0\right), \qquad u_2 = \left(0,\ a(s)\int_0^s a^{-3}(t)\,dt\right).$$

All of these eigenvalues are geometrically simple.

(ii) *There exist real constants $C, \xi_0 > 0$ such that*

$$\|u\|_Y \le C\|(L - i\xi I)u\|_X, \qquad \|u\|_X \le \frac{C}{|\xi|}\|(L - i\xi I)u\|_X$$

for all $u \in Y$ and real ξ with $|\xi| > \xi_0$.

Proof. Part (i) follows from an analysis of the Sturm–Liouville problem (4.18), see [18]. To prove part (ii), we argue as in Lemma 3.4 of [17]. Let $(w, \theta) \in \mathscr{D}(L)$ and set $(f, g) = (L - i\xi)(w, \theta)$. Then

$$a^{-1}\theta - i\xi w = f, \qquad -(a^3 w_s)_s - i\xi\theta = g, \tag{4.19}$$

$$-a^3 w_s(1) + \alpha w(1) = 0, \qquad w(0) = \theta(0) = 0. \tag{4.20}$$

Rewriting $a^3|f_s|^2 + a^{-1}|g|^2$ using (4.19), integrating by parts, and then using (4.20) we obtain

$$C_1(\|f\|_{H^1}^2 + \|g\|_{L^2}^2) \geq \|w\|_{H^2}^2 + \|\theta\|_{H^1}^2 + |\xi|^2(\|w\|_{H^1}^2 + \|\theta\|_{L^2}^2) - C_2|\xi||w_s(1)\theta(1)|. \tag{4.21}$$

It remains only to treat the rightmost term in (4.21). By (4.20), $|w_s(1)| \leq C\|w_s\|_{L^2}$. To estimate $|\theta(1)|$, we let $\varepsilon \in (0,1)$ and use (4.19) to get

$$(a^{-1}(1)\theta(1))^2 = \int_0^1 2(a^{-1}\theta)_s(a^{-1}\theta)\,ds \leq \varepsilon^2|\xi|^2\|w_s\|_{L^2}^2 + \varepsilon^2\|f_s\|_{L^2}^2 + \frac{C}{\varepsilon^2}\|\theta\|_{L^2}^2.$$

The statement then follows by first taking ε sufficiently small and then $|\xi|$ sufficiently large. ∎

Let $X^{\mathrm{c}} \subset X$ be the two-dimensional "center" subspace associated with the eigenvalue 0 of L, and let P^{c} be the spectral projection onto X^{c}. Writing $P^{\mathrm{su}} = (I - P^{\mathrm{c}})$ and $X^{\mathrm{su}} = P^{\mathrm{su}}X$, we decompose X as $X = X^{\mathrm{c}} \oplus X^{\mathrm{su}}$. Here X^{c} and the infinite-dimensional space X^{su} are both invariant subspaces of L. We note that L is not self-adjoint. Indeed, the eigenvalue 0 has algebraic multiplicity 2 but geometric multiplicity 1. Thus the projection P^{c} is not guaranteed to be orthogonal.

Next we turn to the inhomogeneous linear problem $v_x - Lv = g$. Because L has 0 as an eigenvalue with algebraic multiplicity 2, we allow v and g to grow exponentially with a small constant and specify the two-dimensional initial condition $P^{\mathrm{c}}v(0)$. This exponential-dichotomy type result is explained in Lemma 4.10 below.

For any Hilbert space E, $\nu \geq 0$, and $f\colon \mathbb{R} \to E$, we define the norms

$$\|f\|_{L^2_\nu(\mathbb{R},E)} = \|e^{-\nu|\cdot|}f\|_{L^2(\mathbb{R},E)}, \quad \|f\|_{H^1_\nu(\mathbb{R},E)} = \|e^{-\nu|\cdot|}f\|_{L^2(\mathbb{R},E)} + \|e^{-\nu|\cdot|}f_x\|_{L^2(\mathbb{R},E)},$$

and the corresponding spaces

$$L^2_\nu(\mathbb{R},E) = \{f : \|f\|_{L^2_\nu(\mathbb{R},E)} < \infty\}, \quad H^1_\nu(\mathbb{R},E) = \{f \in L^2_\nu(\mathbb{R},E) : f_x \in L^2_\nu(\mathbb{R},E)\}.$$

LEMMA 4.10. *If $\nu > 0$ is sufficiently small, then the linear system*

$$v_x - Lv = g, \qquad P^{\mathrm{c}}v(0) = \eta \tag{4.22}$$

has a unique solution $v \in H^1_\nu(\mathbb{R},X) \cap L^2_\nu(\mathbb{R},\mathscr{D}(L))$ for all $g \in L^2_\nu(\mathbb{R},X)$ and $\eta \in X^{\mathrm{c}}$. Moreover,

$$\|v\|_{L^2_\nu(\mathbb{R},Y)} + \|v\|_{H^1_\nu(\mathbb{R},X)} \leq C(\|\eta\|_X + \|g\|_{L^2_\nu(\mathbb{R},X)}),$$

where the constant C depends only on ν and L.

Proof. Using the decomposition $X = X^{\mathrm{c}} \oplus X^{\mathrm{su}}$, we write $v = (v^{\mathrm{c}}, v^{\mathrm{su}})$, $g = (g^{\mathrm{c}}, g^{\mathrm{su}})$, and $L = (L^{\mathrm{c}}, L^{\mathrm{su}})$, where $L^{\mathrm{c}} = L|_{X^{\mathrm{c}}}\colon X^{\mathrm{c}} \to X^{\mathrm{c}}$ and similarly for L^{su}. Then (4.22) can be written as two decoupled equations

$$v_x^{\mathrm{su}} - L^{\mathrm{su}}v^{\mathrm{su}} = g^{\mathrm{su}}, \tag{4.23}$$

$$v_x^{\mathrm{c}} - L^{\mathrm{c}}v^{\mathrm{c}} = g^{\mathrm{c}}, \qquad v^{\mathrm{c}}(0) = \eta. \tag{4.24}$$

The first equation, (4.23), is an infinite-dimensional equation whose linear operator L^{su} has its spectrum bounded away from the imaginary axis. We claim that for $\nu > 0$

sufficiently small, the unique solution $v^{\mathrm{su}} \in H^1_\nu(\mathbb{R}, X^{\mathrm{su}}) \cap L^2_\nu(\mathbb{R}, \mathscr{D}(L^{\mathrm{su}}))$ of (4.23) satisfies

$$\|v^{\mathrm{su}}\|_{H^1_\nu(\mathbb{R},X)} + \|v^{\mathrm{su}}\|_{L^2_\nu(\mathbb{R},\mathscr{D}(L))} \le C\|g^{\mathrm{su}}\|_{L^2_\nu(\mathbb{R},X)}.$$

Thanks to the bounds in Lemma 4.9, the claim with $\nu = 0$ follows by taking a Fourier transform in x; the estimate for $\nu > 0$ is then obtained by a simple perturbation argument; see, for instance, [35, 36]. The second equation, (4.24), is a two-dimensional linear system whose linear operator L^{c} is a 2×2 Jordan block with eigenvalue 0. Thus an elementary argument shows that the solution v^{c} of (4.24) satisfies

$$\|v^{\mathrm{c}}\|_{H^1_\nu(\mathbb{R},X^{\mathrm{c}})} \le C(\|\eta\|_X + \|g^{\mathrm{c}}\|_{L^2_\nu(\mathbb{R},\mathbb{R}^n)}),$$

where the constant C again depends only on ν and L^{c}. The lemma then follows from combining the above results for (4.23) and (4.24). $\square$

The following lemma extends Lemma 4.10 to nonautonomous perturbations of L.

LEMMA 4.11. *Suppose $M(x)\colon \mathscr{D}(L) \to X$ is a family of bounded linear operators depending on $x \in \mathbb{R}$, and that $\|M(x)\|_{\mathscr{D}(L)\to X} \le \varepsilon$ for all x. If ε is sufficiently small, then the nonautonomous linear system*

$$v_x - Lv = M(x)v, \qquad P^{\mathrm{c}}v(0) = \eta \tag{4.25}$$

has a unique solution $v \in H^1_\nu(\mathbb{R}, X) \cap L^2_\nu(\mathbb{R}, \mathscr{D}(L))$ for each $\eta \in X^{\mathrm{c}}$.

Proof. We first consider the inhomogeneous system

$$v_x - Lv = M(x)\varphi, \qquad P^{\mathrm{c}}v(0) = \eta \tag{4.26}$$

with $\varphi \in L^2_\nu(\mathbb{R}, \mathscr{D}(L))$. By our assumption on M,

$$\|M(x)\varphi\|_{L^2_\nu(\mathbb{R},X)} \le \varepsilon\|\varphi\|_{L^2_\nu(\mathbb{R},\mathscr{D}(L))}. \tag{4.27}$$

Setting $g = M(x)\varphi$ in Lemma 4.10, we see that for each $g \in L^2_\nu(\mathbb{R}, \mathscr{D}(L))$, (4.26) has a unique solution $v \in L^2_\nu(\mathbb{R}, \mathscr{D}(L)) \cap H^1_\nu(\mathbb{R}, X)$. Denoting this v by $T^\eta(\varphi)$, (4.25) becomes the fixed point equation $v = T^\eta(v)$. Lemma 4.10 and (4.27) give an estimate

$$\|T^\eta(\varphi)\|_{L^2_\nu(\mathbb{R},\mathscr{D}(L))} \le C(\|\eta\|_X + \varepsilon\|\varphi\|_{L^2_\nu(\mathbb{R},\mathscr{D}(L))}),$$

where the constant C is independent of η. The identity $T^\eta(w_1) - T^\eta(w_2) = T^0(w_1 - w_2)$ for w_1, w_2 in $L^2_\nu(\mathbb{R}, \mathscr{D}(L))$ then yields

$$\|T^\eta(w_1) - T^\eta(w_2)\|_{L^2_\nu(\mathbb{R},\mathscr{D}(L))} \le C\varepsilon\|w_1 - w_2\|_{L^2_\nu(\mathbb{R},\mathscr{D}(L))}.$$

Picking $\varepsilon < 1/C$, T^η is therefore a uniform contraction $L^2_\nu(\mathbb{R}, \mathscr{D}(L)) \to L^2_\nu(\mathbb{R}, \mathscr{D}(L))$, and hence (4.25) has a unique solution $v \in L^2_\nu(\mathbb{R}, \mathscr{D}(L))$. Combining this with the equation $v_x = Lv + M(x)v$, we have $v \in H^1_\nu(\mathbb{R}, X)$. $\square$

4.3. Center manifold reduction. In this section, we will describe the two-dimensional center manifold M^ζ constructed in [18]. This manifold contains all small bounded solutions u of $u_x = Lu + N^\zeta(u)$, and in particular all small-amplitude solitary waves.

Let $U \subset \mathscr{D}(L)$ and Λ be the neighborhoods of the origin from Lemma 4.2, and let $U^{\mathrm{c}} = P^{\mathrm{c}}U$ be the projection of U onto the two-dimensional space X^{c}. We also fix a basis $\{e_1, e_2\}$ of X^{c} given by $e_1 = d_1^{-1} u_1$, $e_2 = d_1^{-1} u_2$, where u_1, u_2 are the generalized eigenvectors from Lemma 4.9 and

$$d_1^2 = \int_0^1 a(s) \left(\int_0^s a^{-3}(t)\, dt \right) ds$$

is a constant. The results of the center manifold construction in [18] that we need are summarized by the following lemma.

LEMMA 4.12. *Fix an integer $k \geq 2$. After possibly shrinking Λ and U, there exists, for each $\zeta \in \Lambda$, a two-dimensional manifold $M^\zeta \subset U$ with an invertible coordinate map $\chi^\zeta \colon M^\zeta \to U^{\mathrm{c}}$ satisfying the following properties:*

(i) *Every initial condition $u_0 \in M^\zeta$ determines a unique solution u of $u_x = Lu + N^\zeta(u)$ which remains in M^ζ as long as it remains in U.*

(ii) *If u solves $u_x = Lu + N^\zeta(u)$ and satisfies $u(x) \in U$ for all x, then u lies entirely in M^ζ.*

(iii) *Defining $r^\zeta \colon U^{\mathrm{c}} \to U$ by*

$$u^{\mathrm{c}} + r^\zeta(u^{\mathrm{c}}) = (\chi^\zeta)^{-1}(u^{\mathrm{c}}), \tag{4.28}$$

the map $(\zeta, u) \mapsto r^\zeta(u)$ is $C^k(\Lambda \times U^{\mathrm{c}}, U)$. Moreover, $r^\zeta(0) = 0$ for all $\zeta \in \Lambda$ and $Dr^0(0) = 0$.

(iv) *If $u^{\mathrm{c}} \in C^1((a,b), U^{\mathrm{c}})$ solves the reduced system*

$$u^{\mathrm{c}}_x = f^\zeta(u^{\mathrm{c}}) := D\chi^\zeta(u)(Lu + N^\zeta(u)), \qquad \text{where } u = u^{\mathrm{c}} + r^\zeta(u^{\mathrm{c}}), \tag{4.29}$$

then $u = u^{\mathrm{c}} + r^\zeta(u^{\mathrm{c}})$ solves the full equation $u_x - Lu = N^\zeta(u)$.

(v) *The two-dimensional system (4.29) is reversible. Writing $u^{\mathrm{c}} \in U^{\mathrm{c}}$ as $u^{\mathrm{c}} = qe_1 + pe_2$ and setting $c_0 = \alpha_{\mathrm{cr}}^3 \int_0^1 a^{-5}(s)\, ds$, we have*

$$q_x = p + R_1(q, p; \zeta), \qquad p_x = \frac{\zeta}{\alpha_{\mathrm{cr}}^2 d_1^2} q - \frac{3c_0}{2\alpha_{\mathrm{cr}}^3 d_1^3} q^2 + R_2(q, p; \zeta), \tag{4.30}$$

where the C^k error terms R_1, R_2 are odd and even in p, respectively, and satisfy the bounds

$$\begin{aligned} R_1 &= O\big(|(q,p)| \cdot |(\zeta, q, p)|\big), \\ R_2 &= O\big(|p| \cdot |(\zeta, q, p)|\big) + O\big(|q,p| \cdot |(\zeta, q, p)|^2\big). \end{aligned} \tag{4.31}$$

The action of the reverser in these coordinates is $(q, p) \mapsto (q, -p)$.

Proof. Since this is shown in [18], we only give a brief outline. The first step is to apply Theorem 3.1 in [18], which is a parametrized, Hamiltonian version of a reduction principle for quasilinear evolution equations originally due to Mielke [36], making use of the Hamiltonian structure of $u_x - Lu = N^\zeta(u)$ and Lemma 4.9. This gives a center manifold M^ζ with a coordinate map $\tilde{\chi}^\zeta \colon M^\zeta \to U^{\mathrm{c}}$ satisfying conditions (i)–(iv). Several changes of coordinates are then performed, which finally allow (4.30) to be obtained by Taylor expansion. ∎

Lemma 4.12 has the following easy corollary concerning linearized problems.

COROLLARY 4.13. *Let u, u^{c} be as in* (iv) *of Lemma 4.12. If $v^{\mathrm{c}} \in C^1(\mathbb{R}, U^{\mathrm{c}})$ solves the linearized reduced equation $v^{\mathrm{c}}_x = Df^\zeta(u^{\mathrm{c}}) v^{\mathrm{c}}$, then $v = v^{\mathrm{c}} + Dr^\zeta(u^{\mathrm{c}}) v^{\mathrm{c}}$ solves the full linearized equation $v_x - Lv = DN^\zeta(u) v$.*

Proof. The statement follows from plugging $u = u^{\mathrm{c}} + r^\zeta(u^{\mathrm{c}})$ into the full equation $u_x - Lu = N^\zeta(u)$ and differentiating both sides with respect to u^{c}. ∎

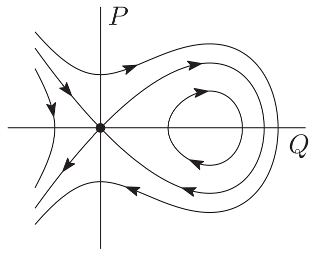

FIG. 2. *Phase portrait of the rescaled system* (4.33) *with* $\zeta > 0$ *small.*

4.4. Existence and uniqueness of small-amplitude solutions. We now construct homoclinic orbits u^ζ of $u_x - Lu = N^\zeta(u)$ and solutions v^ζ of the associated linearized problems by lifting solutions of the reduced equations.

LEMMA 4.14. *There exists* $\zeta_* > 0$ *such that* $u_x - Lu = N^\zeta(u)$ *has a reversible homoclinic orbit* u^ζ *for* $0 < \zeta < \zeta_*$ *with the following properties:*

(i) u^ζ *satisfies an exponential bound*

$$\sum_{k=0}^{3} \|D^k u^\zeta(x)\|_{\mathscr{D}(L)} \le C\zeta e^{-c\sqrt{\zeta}\cdot|x|}$$

for some positive constants C, c *independent of* x *and* ζ.

(ii) *There exists a solution* v^ζ *of the linearized problem* $v_x^\zeta = Lv^\zeta + DN^\zeta(u^\zeta)v^\zeta$ *which is linearly independent from* u_x^ζ, *unbounded in* Y *as* $x \to \pm\infty$, *and satisfies the exponential growth estimate* $\|v^\zeta(x)\|_Y \le Ce^{+c\sqrt{\zeta}\cdot|x|}$ *for some positive constants* C, c *independent of* x *and* ζ.

(iii) *The map* $\zeta \mapsto u^\zeta$ *is continuous from the interval* $(0, \zeta_*)$ *to* $C^3_\sigma(\mathbb{R}, X) \cap C^2_\sigma(\mathbb{R}, \mathscr{D}(L))$.

Proof. In the notation of part (v) of Lemma 4.12, we introduce, for $\zeta > 0$, the scaled variables

$$X = \frac{\sqrt{\zeta}}{\alpha_{\mathrm{cr}} d_1} x, \qquad q(x) = \frac{\alpha_{\mathrm{cr}} d_1}{c_0} \zeta Q(X), \qquad p(x) = \frac{\zeta^{3/2}}{c_0} P(X), \tag{4.32}$$

so that (4.30) becomes

$$Q_X = P + R_3(Q, P; \zeta), \qquad P_X = Q - \frac{3}{2} Q^2 + R_4(Q, P; \zeta), \tag{4.33}$$

where R_3 and R_4 are $O(\zeta^{1/2})$ and, respectively, odd and even in P. Sending $\zeta \to 0$ in (4.33) we're left with the system $Q_X = P$, $P_X = Q - \frac{3}{2} Q^2$, which has a reversible homoclinic orbit $Q^0(X) = \operatorname{sech}^2(X/2)$. Exploiting reversibility as in section 4.1 of [17], we conclude that the phase portrait of (4.33) is qualitatively the same for $\zeta > 0$ sufficiently small, say $0 < \zeta < \zeta_*$; see Figure 2. In particular, (4.33) has a reversible homoclinic orbit (Q^ζ, P^ζ) with $Q^\zeta > 0$. Since $(Q^\zeta, P^\zeta)(0)$ and the local stable and unstable manifolds of (4.33) at $(0, 0)$ depend continuously on ζ, we have uniform bounds

$$\sum_{k=0}^{3} |D_X^k (Q^\zeta, P^\zeta)| \le Ce^{-\frac{1}{2}|X|}.$$

Defining (q^ζ, p^ζ) in terms of (Q^ζ, P^ζ) by (4.32), we therefore have a reversible homoclinic orbit $u^{\zeta,\mathrm{c}} = q^\zeta e_1 + p^\zeta e_2$ of (4.29) with the similar exponential bound $\sum_{k=0}^3 |D_x^k u^{\zeta,\mathrm{c}}| \le C_1 \zeta e^{-C_2\sqrt{\zeta}|x|}$ for some positive constants C_1, C_2. Since our weight function σ grows more slowly than any exponential, the continuity of $\zeta \mapsto u^{\zeta,\mathrm{c}}$ from the interval $(0,\zeta_*)$ to $C^3_\sigma(\mathbb{R}, U^\mathrm{c})$ then follows from the continuity of $\zeta \mapsto u^{\zeta,\mathrm{c}}(0)$.

Now we consider the linearized reduced equation

$$v^\mathrm{c}_x = Df^\zeta(u^{\zeta,\mathrm{c}})v^\mathrm{c}. \tag{4.34}$$

Differentiating (4.29), we get as usual that $v^\mathrm{c} = u^{\zeta,\mathrm{c}}_x$ is a solution of (4.34). Since (4.34) is a two-dimensional system, it has one other linearly independent solution, which we denote by $v^{\zeta,\mathrm{c}}$. Looking at (4.30)–(4.31), we see that $Df^\zeta(0)$ has eigenvalues $\pm\sqrt{\zeta} + O(\zeta)$ corresponding to eigenvectors $e_1 \pm \sqrt{\zeta} e_2 + O(\zeta)$. By an elementary argument (for instance, Problem 29 in Chapter 3 of [8]), $v^{\zeta,\mathrm{c}}$ is unbounded as $x \to \pm\infty$ with

$$|v_2^{\zeta,\mathrm{c}}(x)| \le C_1 e^{+C_2\sqrt{\zeta}\cdot|x|},$$

where the constants $C_1, C_2 > 0$ can be chosen independently of ζ. Define

$$u^\zeta = u^{\zeta,\mathrm{c}} + r^\zeta(u^{\zeta,\mathrm{c}}), \qquad v^\zeta = v^{\zeta,\mathrm{c}} + Dr^\zeta(u^{\zeta,\mathrm{c}})v^{\zeta,\mathrm{c}}.$$

By Lemma 4.12(iv), u^ζ is a reversible homoclinic orbit of the full system $u_x - Lu = N^\zeta(u)$, and by Corollary 4.13, v^ζ is a solution of the full linearized system $v_x - Lv = DN^\zeta(u)v$, linearly independent from u^ζ_x. Thanks to the properties of r^ζ from Lemma 4.12(iii), our exponential estimates for $u^{\zeta,\mathrm{c}}, v^{\zeta,\mathrm{c}}$ carry over to u^ζ, v^ζ, after possibly shrinking ζ_*. Similarly the continuity of $\zeta \mapsto u^{\zeta,\mathrm{c}}$ in $C^3_\sigma(\mathbb{R}, U^\mathrm{c})$ implies the continuity of $\zeta \mapsto u^\mathrm{c}$ in $C^3_\sigma(\mathbb{R}, X) \cap C^2_\sigma(\mathbb{R}, \mathscr{D}(L))$. ∎

Combining Lemma 4.14 with Lemma 4.11, we now show that any solution of $v_x - Lv = DN^\zeta(u^\zeta)v$ with at most mildly exponential growth as $x \to \pm\infty$ must be a linear combination of v^ζ, u^ζ_x.

Lemma 4.15. *Let $0 < \zeta < \zeta_*$ and u^ζ, v^ζ be as in Lemma 4.14, and let $\nu > 0$ be as in Lemma 4.10. After possibly shrinking ζ_*, the space of solutions v to*

$$v_x - Lv = DN^\zeta(u^\zeta)v, \tag{4.35}$$

$$v \in H^1_\nu(\mathbb{R}, X) \cap L^2_\nu(\mathbb{R}, \mathscr{D}(L)) \tag{4.36}$$

is two-dimensional and spanned by v^ζ, u^ζ_x.

Proof. Set $M(x) = DN^\zeta(u^\zeta(x))$. We first claim that, after possibly shrinking ζ_*, the system

$$v_x - Lv = M(x)v, \qquad P^\mathrm{c}v(0) = \eta,$$

has a unique solution $v \in L^2_\nu(\mathbb{R}, \mathscr{D}(L)) \cap H^1_\nu(\mathbb{R}, X)$ for each $\eta \in X^\mathrm{c}$. We know that $N^\zeta : U \to X$ is smooth with $DN^0(0) = 0$, and Lemma 4.14 gives $\|u^\zeta(x)\|_{\mathscr{D}(L)} \le C|\zeta|$. Combining these facts we have

$$\|M(x)\|_{\mathscr{D}(L)\to X} = \|DN^\zeta(u^\zeta(x))\|_{\mathscr{D}(L)\to X} \le C(\|u^\zeta(x)\| + \zeta) \le C\zeta$$

for all x. Picking ζ_* sufficiently small, the claim then follows from Lemma 4.11.

Since X^c is two-dimensional, the space of solutions v to (4.35)–(4.36) is also two-dimensional. Thus, to prove the lemma, it suffices to show that v^ζ, u^ζ_x are linearly independent solutions of (4.35)–(4.36). From Lemma 4.14 we know that v^ζ, u_x^ζ are linearly independent solutions of (4.35), so all that remains to show is the integrability $v^\zeta, u_x^\zeta \in L^2_\nu(\mathbb{R}, \mathscr{D}(L)) \cap H^1_\nu(\mathbb{R}, X)$. By part (i) of Lemma 4.14, u_x^ζ and u_{xx}^ζ decay exponentially in $\mathscr{D}(L)$ as $x \to \pm\infty$, so in particular $u_x^\zeta \in H^1_\nu(\mathbb{R}, \mathscr{D}(L))$. Now we consider v^ζ. By part (ii) of Lemma 4.14 we have $\|v^\zeta(x)\|_{\mathscr{D}(L)} \le Ce^{+c\sqrt{\zeta}\cdot|x|}$, where $c > 0$ is independent of ζ. Thus $v^\zeta \in L^2_\nu(\mathbb{R}, \mathscr{D}(L))$ as long as $c|\zeta_*|^{1/2} < \nu$. Finally, the equation $v_x^\zeta = Lv^\zeta + M(x)v^\zeta$ implies $v^\zeta \in H^1_\nu(\mathbb{R}, X)$. $\square$

Finally, we prove Theorem 4.1 by combining Lemmas 4.3, 4.4, 4.14, and 4.15.

Proof of Theorem 4.1. Let $\zeta_*, u^\zeta, v^\zeta$ be as in Lemmas 4.14 and 4.15. Combining Lemma 4.3 with Lemma 4.14 we see that u^ζ corresponds to a nontrivial solution $w^\zeta \in \mathscr{X}_\sigma^{\mathrm{e}}$ of $\mathscr{F}(\zeta, w) = 0$, depending continuously on $0 < \zeta < \zeta_*$ and with $|w^\zeta|_{2+\beta} \to 0$ as $\zeta \to 0$, which is (i).

Next we show (ii), that the linearized operators $\mathscr{F}_w(\zeta, w^\zeta)$ are invertible for $0 < \zeta < \zeta_*$. By Corollary 3.5, $\mathscr{F}_w(\zeta, w^\zeta)$ is Fredholm of index 0, so it suffices to show that it has trivial kernel. Fix $0 < \zeta < \zeta_*$ and assume for contradiction that $0 \ne \varphi \in \mathscr{X}_\sigma^{\mathrm{e}}$ satisfies $\mathscr{F}_w(\zeta, w^\zeta)\varphi = 0$. Let $v \in C^1_{\mathrm{b}}(\mathbb{R}, X) \cap C^0_{\mathrm{b}}(\mathbb{R}, \mathscr{D}(L))$ be the corresponding nontrivial solution of $v_x - Lv = DN^\zeta(u^\zeta)v$ given by Lemma 4.4, and recall that $v(-x) = Sv(x)$. Pick $\nu > 0$ as in Lemma 4.15. Since $v \in H^1_\nu(\mathbb{R}, X) \cap L^2_\nu(\mathbb{R}, \mathscr{D}(L))$, we have by Lemma 4.15 that v is a linear combination of v^ζ, u_x^ζ. Since $\|v^\zeta(x)\|_Y$ is unbounded by Lemma 4.14(ii), v must be a scalar multiple of u_x^ζ. Differentiating $u^\zeta(-x) = Su^\zeta(x)$ we discover $u_x^\zeta(-x) = -Su_x^\zeta(x)$, and hence $v(-x) = -Sv(x)$. But we already know $v(-x) = Sv(x)$, so this forces $Sv(x) \equiv 0$ and hence $v \equiv 0$.

Now we show (iii). Let (ζ, w) be a nontrivial solution of $\mathscr{F}(\zeta, w) = 0$ with $\zeta > 0$ and $|\zeta| + |w|_2 < \delta$, and assume for contradiction that $w \ne w^\zeta$. By the monotonicity of w and w^ζ (Proposition 2.2), w is not a translate of w^ζ. If δ is sufficiently small, Lemma 4.12 implies that

$$u = G^\zeta\left(w, \frac{w_x}{H_s + w_s}\right) = qe_1 + pe_2 + r^\zeta(qe_1 + pe_2),$$

where $u^{\mathrm{c}} = qe_1 + pe_2$ is a reversible homoclinic orbit of the two-dimensional reduced equation, (4.29). Moreover, u^{c} is not a translate of $u^{\zeta,\mathrm{c}}$, where $u^{\zeta,\mathrm{c}} = q^\zeta e_1 + p^\zeta e_2$ is the solution of the reduced equation associated to w^ζ. Tracing back the various changes of variable, we have

$$w(x, s) = q(x)e_1(s) + R(x, s),$$

where the remainder term R satisfies

$$\|R(x, \cdot)\|_{H^2(0,1)} \le C\big(|\zeta| + |q| + |p|\big)\big(|q| + |p|\big)$$

with the constant C independent of ζ. We take δ small enough that the above estimate implies

$$|R(x, 1)| \le \frac{e_1(1)}{2}\big(|q(x)| + |p(x)|\big). \tag{4.37}$$

Now we analyze the reduced system (4.29) near the saddle point 0. Since (4.29) already has one homoclinic orbit $u^{\zeta,\mathrm{c}}$, we can find the angles at which u^{c} approaches 0 as $x \to \pm\infty$; see Figure 2. As mentioned in the proof of Lemma 4.14, $Df^\zeta(0)$ has eigenvalues $\pm\sqrt{\zeta} + O(\zeta)$ corresponding to the eigenvectors $e_1 \pm \sqrt{\zeta}e_2 + O(\zeta)$. Thus we have $q < 0$ for $|x|$ sufficiently large, and

$$\lim_{x\to\pm\infty} \frac{p}{q} = \mp\sqrt{\zeta} + O(\zeta). \tag{4.38}$$

Further shrinking δ, (4.37) and (4.38) imply

$$|R(x,1)| \le \frac{3e(1)|q(x)|}{4} < e(1)|q(x)| \tag{4.39}$$

for $|x|$ sufficiently large. Thus, for $|x|$ large enough that $q < 0$ and (4.39) both hold,

$$w(x,1) = q(x)e_1(1) + R(x,1) < 0.$$

But (ζ, w) is a nontrivial supercritical solitary wave, so $w(x,1) > 0$ by Proposition 2.1, a contradiction. $\square$

5. Global continuation. In this section we will prove Theorem 1.3. Our first step in this direction is Theorem 5.2, which we will prove in sections 5.1–5.4 using a topological degree argument. Given $\delta > 0$ and an arbitrary weight function σ satisfying the assumptions of section 3.1, Theorem 5.2 asserts that a certain subset $\mathscr{C}_\sigma^{\delta,+}$ of $\mathscr{C}$ is either unbounded in $\mathbb{R}\times\mathscr{X}_\sigma^{\mathrm{e}}$ or contains solutions with $\zeta = \delta$ or $\zeta = \alpha_{\mathrm{cr}} - \delta$. In section 5.1, we will define the weighted continuum $\mathscr{C}_\sigma^\delta$, which is a connected subset of $\mathbb{R}\times\mathscr{X}_\sigma^{\mathrm{e}}$ containing solutions with $\delta \le \zeta \le \alpha_{\mathrm{cr}} - \delta$. In section 5.2, we will use the invertibility results from sections 3–4 to show that removing a point in $\mathscr{C}_{\mathrm{loc}}$ from $\mathscr{C}_\sigma^\delta$ splits it into exactly two components, the more interesting of which is $\mathscr{C}_\sigma^{\delta,+}$. In section 5.3, we will use the results of section 3 to define the Healey–Simpson degree for our nonlinear operator $\mathscr{F}$. We will then use this degree in section 5.4 to prove Theorem 5.2, again using the invertibility from section 4.

In sections 5.5–5.7, we will prove Theorem 1.3 by analyzing the alternatives in Theorem 5.2 as $\delta \to 0$. If $\mathscr{C}_\sigma^{\delta,+}$ is unbounded, then, since we always have $0 < \zeta < \alpha_{\mathrm{cr}}$, there must be a sequence in $\mathscr{C}_\sigma^{\delta,+}$ with $|\sigma w_n|_{2+\beta} \to \infty$. In section 5.5, we will reduce this condition to $|\sigma w_n|_0 + |\partial_s w_n|_0 \to \infty$. To accomplish this we will use the lower bound on the pressure in Lemma 2.9, the weighted Schauder estimates from Appendix A, and regularity results of Lieberman [32] for fully nonlinear elliptic problems. In section 5.6 we will use the equidecay result from section 2 to *construct a particular weight function* σ for which this condition further reduces to $|\partial_s w_n|_0 \to \infty$. This yields (i) in Theorem 1.3. In section 5.7, we will send $\delta \to 0$ and address the remaining possibilities, that there exist solutions $(\zeta_n, w_n) \in \mathscr{C}$ with $\zeta_n \to 0$ or $\zeta_n \to \alpha_{\mathrm{cr}}$. For ζ near α_{cr}, we will obtain alternative (ii) of Theorem 1.3 by using the upper bound Proposition 2.3 on the Froude number. Finally, for ζ_n near 0, we will use the uniqueness result from section 4 to obtain alternative (iii).

5.1. The weighted continuum. We first recall Definition 1.2. The set $\mathscr{S}$ of supercritical solitary waves is

$$\mathscr{S} = \{(\zeta, w) : (\zeta, w) \text{ satisfies (1.19)}, 0 < \zeta < \alpha_{\mathrm{cr}}\},$$

which we view as a subset of $\mathbb{R}\times C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$, and the global continuum $\mathscr{C}$ is the connected component of $\mathscr{S}$ in $\mathbb{R}\times C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$ containing $\mathscr{C}_{\mathrm{loc}}$. Here $\mathscr{C}_{\mathrm{loc}}$ is the local curve of nontrivial solutions given by Theorem 4.1,

$$\mathscr{C}_{\mathrm{loc}} = \{(\zeta, w^\zeta) : 0 < \zeta < \zeta_*\}, \quad \text{where } |w^\zeta|_{2+\beta} \to 0 \text{ as } \zeta \to 0.$$

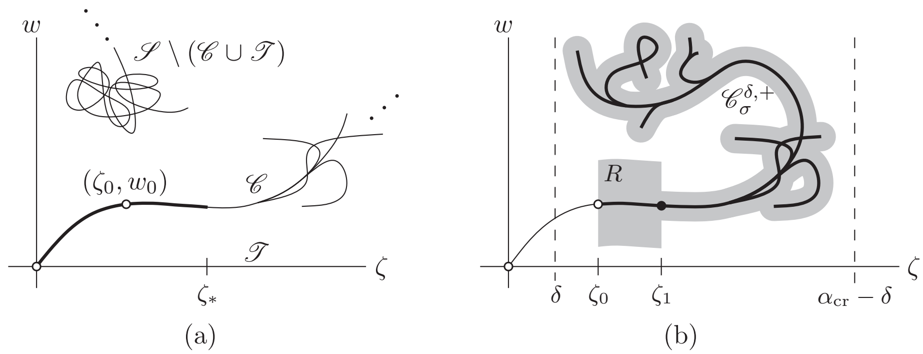

FIG. 3. (a) *The components of* $\mathscr{S} \setminus \{(\zeta_0, w_0)\}$ *in Lemma* 5.3. *The bold portion of* $\mathscr{C}$ *is* $\mathscr{C}_{\mathrm{loc}}$. (b) *The neighborhood* U *in the proof of Theorem* 5.2. U *is the shaded region,* R *is the strip with* $\zeta_0 < \zeta < \zeta_1$, *and* $\mathscr{C}_\sigma^{\delta,+}$ *is drawn in bold.*

We now define analogues of $\mathscr{S}$ and $\mathscr{C}$ in weighted spaces. Fix a weight function σ as in section 3.1, and define the weighted spaces $\mathscr{X}_\sigma^{\mathrm{e}}, \mathscr{Y}_\sigma^{\mathrm{e}}$ as in (3.2), (3.5). Recall the open subset $\mathscr{U}_\sigma \subset \mathscr{X}_\sigma^{\mathrm{e}}$ defined in (3.3),

$$\mathscr{U}_\sigma = \left\{ w \in \mathscr{X}_\sigma^{\mathrm{e}} : \inf_\Omega (H_s + w_s) > \frac{\delta_*}{2} \right\},$$

where $\delta_* > 0$ is fixed and given by Proposition 2.4. As in section 3.4, we introduce a small parameter $0 < \delta < \zeta_*$ and work with (ζ, w) in the set $[\delta, \alpha_{\mathrm{cr}} - \delta] \times \overline{\mathscr{U}_\sigma}$, where the usual inequalities $0 < \zeta < \alpha_{\mathrm{cr}}$ hold uniformly. For convenience, we shrink ζ_* so that $2\zeta_* < \alpha_{\mathrm{cr}}$.

DEFINITION 5.1 (weighted continuum). *For* $0 < \delta < \zeta_*$, *define*

$$\mathscr{S}_\sigma^\delta := \mathscr{S} \cap \left([\delta, \alpha_{\mathrm{cr}} - \delta] \times \overline{\mathscr{U}_\sigma} \right) \subset \mathbb{R} \times C_\sigma^{2+\beta}(\overline{\Omega}).$$

The weighted continuum $\mathscr{C}_\sigma^\delta$ *is the connected component of* $\mathscr{S}_\sigma^\delta$ *in* $\mathbb{R} \times C_\sigma^{2+\beta}(\overline{\Omega})$ *containing* $\mathscr{C}_{\mathrm{loc}} \cap \mathscr{S}_\sigma^\delta$.

We are now ready to state the first main result of this section.

THEOREM 5.2 (global continuation). *Fix* $0 < \delta < \zeta_*$ *and* $(\zeta_0, w_0) \in \mathscr{C}_{\mathrm{loc}} \cap \mathscr{C}_\sigma^\delta$. *Then* $\mathscr{C}_\sigma^\delta \setminus \{(\zeta_0, w_0)\}$ *has exactly two connected components. One component is* $\mathscr{C}_{\mathrm{loc}} \cap \{\delta \le \zeta < \zeta_0\}$, *and the other component* $\mathscr{C}_\sigma^{\delta,+}$ *is either unbounded or meets the boundary of* $(\delta, \alpha_{\mathrm{cr}} - \delta) \times \mathscr{U}_\sigma$.

5.2. Connectedness properties. In this section we will prove the disconnectedness statement in Theorem 5.2 using the implicit function theorem and the invertibility of $\mathscr{F}_w(\zeta, w)$ given by Corollary 3.4 and part (ii) of Theorem 4.1. Our first lemma asserts that $\mathscr{S}$ has at least two components and that removing a point in $\mathscr{C}_{\mathrm{loc}}$ splits it into at least three; see Figure 3(a).

LEMMA 5.3. *Let* $\mathscr{T} = \{(\zeta, 0) : 0 < \zeta < \alpha_{\mathrm{cr}}\}$ *denote the set of trivial solutions in* $\mathscr{S}$. *Then* $\mathscr{T}$ *is a connected component of* $\mathscr{S}$. *In particular, since connected components are disjoint,* $\mathscr{C} \cap \mathscr{T} = \varnothing$. *Moreover, for each* $(\zeta_0, w_0) \in \mathscr{C}_{\mathrm{loc}}$, $\mathscr{S} \setminus \{(\zeta_0, w_0)\}$ *has at least three connected components, of which the two least interesting components are* $\mathscr{T}$ *and* $\mathscr{C}_{\mathrm{loc}} \cap \{\zeta < \zeta_0\}$. *Here as always we topologize* $\mathscr{S}$ *as a subset of* $\mathbb{R} \times C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$.

Proof. Since we are working in unweighted spaces, it is useful to have an unweighted version of our nonlinear operator $\mathscr{F}$. To make this precise, we set

$$\mathscr{U}_{\mathrm{b}} = \left\{ w \in \mathscr{X}_{\mathrm{b}}^{\mathrm{e}} : \inf_{\Omega}(H_s + w_s) > \frac{\delta_*}{2} \right\},$$

where δ_* is given in Proposition 2.4, and define $\mathscr{F}^{\mathrm{b}}\colon (0,\alpha_{\mathrm{cr}}) \times \mathscr{U}_{\mathrm{b}} \to \mathscr{Y}_{\mathrm{b}}^{\mathrm{e}}$ by the same formula, (3.4), used to define $\mathscr{F}$.

First consider $\mathscr{T}$. Clearly $\mathscr{T}$ is connected (indeed path connected) and relatively closed as a subset of $\mathscr{S}$. It remains to show that $\mathscr{T}$ is relatively open. For each $(\zeta,0) \in \mathscr{T}$, $\mathscr{F}^{\mathrm{b}}(\zeta,0) = 0$. Moreover, $\mathscr{F}_w^{\mathrm{b}}(\zeta,0)$ is invertible $\mathscr{X}_{\mathrm{b}}^{\mathrm{e}} \to \mathscr{Y}_{\mathrm{b}}^{\mathrm{e}}$ by Corollary 3.4. Thus relative openness follows from the implicit function theorem. The same argument shows that $\mathscr{T}$ is a connected component of $\mathscr{S} \setminus \{(\zeta_0,w_0)\}$ for any $(\zeta_0,w_0) \in \mathscr{C}_{\mathrm{loc}}$. Since $\mathscr{C} \subset \mathscr{S}$ is connected and $\mathscr{C} \not\subset \mathscr{T}$, we must have $\mathscr{C} \cap \mathscr{T} = \varnothing$.

Now pick $(\zeta_0,w_0) \in \mathscr{C}_{\mathrm{loc}}$. Since $\zeta \mapsto w^\zeta$ is continuous $(0,\zeta_*) \to \mathscr{X}_{\mathrm{b}}^{\mathrm{e}}$, we easily check that $\mathscr{C}_{\mathrm{loc}} \cap \{\zeta < \zeta_0\}$ is a relatively closed subset of $\mathscr{S} \setminus \{(\zeta_0,w_0)\}$. For any $(\zeta,w^\zeta) \in \mathscr{C}_{\mathrm{loc}}$, $\mathscr{F}^{\mathrm{b}}(\zeta,w^\zeta) = 0$, and by Theorem 4.1, $\mathscr{F}_w^{\mathrm{b}}(\zeta,w^\zeta)$ is invertible $\mathscr{X}_{\mathrm{b}}^{\mathrm{e}} \to \mathscr{Y}_{\mathrm{b}}^{\mathrm{e}}$. Thus we can apply the implicit function theorem as before to deduce that $\mathscr{C}_{\mathrm{loc}} \cap \{\zeta < \zeta_0\}$ is relatively open. The third connected component of $\mathscr{S} \setminus \{(\zeta_0,w_0)\}$ is the one containing $\mathscr{C}_{\mathrm{loc}} \cap \{\zeta > \zeta_0\}$. Of course $\mathscr{S}$ might have connected components other than $\mathscr{C}$ and $\mathscr{T}$, in which case $\mathscr{S} \setminus \{(\zeta_0,w_0)\}$ will have more than three components. ◻

The following elementary lemma asserts that connected subsets of $\mathscr{S}_\sigma^\delta$ are also connected in $\mathscr{S}$.

Lemma 5.4. *Fix $0 < \delta < \zeta_*$. If A is a connected subset of $\mathscr{S}_\sigma^\delta$ in $\mathbb{R} \times C_\sigma^{2+\beta}(\overline{\Omega})$, then A is also a connected subset of $\mathscr{S}$ in $\mathbb{R} \times C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$. In particular, $\mathscr{C}_\sigma^\delta \subset \mathscr{C}$.*

Proof. In what follows, $\mathscr{S}$ is always topologized as a subset of $\mathbb{R} \times C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$, while $\mathscr{S}_\sigma^\delta$ is topologized as a subset of $\mathbb{R} \times C_\sigma^{2+\beta}(\overline{\Omega})$. To prove the first statement, suppose that $A \subset \mathscr{S}_\sigma^\delta$ is disconnected as a subset of $\mathscr{S}$. Then there exist disjoint open sets $B, C \subset \mathbb{R} \times C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$ with $A \cap B \neq \varnothing$, $A \cap C \neq \varnothing$, and $A \subset B \cup C$. Set $B' = B \cap (\mathbb{R} \times C_\sigma^{2+\beta}(\overline{\Omega}))$ and $C' = C \cap (\mathbb{R} \times C_\sigma^{2+\beta}(\overline{\Omega}))$. Then B' and C' are disjoint open subsets of $C_\sigma^{2+\beta}(\overline{\Omega})$ with $A \subset B' \cup C'$, $A \cap B' \neq \varnothing$, and $A \cap C' \neq \varnothing$. Thus A is disconnected as a subset of $\mathscr{S}_\sigma^\delta$. Now we turn to $\mathscr{C}_\sigma^\delta$. By construction, $\mathscr{C}_\sigma^\delta$ is a connected subset of $\mathscr{S}_\sigma^\delta$. By the above argument, it must also be a connected subset of $\mathscr{S}$. Since $\mathscr{C}$ is a connected component of $\mathscr{S}$ and $\mathscr{C}_\sigma^\delta \cap \mathscr{C} \neq \varnothing$, we must have $\mathscr{C}_\sigma^\delta \subset \mathscr{C}$. ◻

Lemma 5.5. *Fix $0 < \delta < \zeta_*$, and pick $(\zeta_0,w_0) \in \mathscr{C}_{\mathrm{loc}} \cap \mathscr{C}_\sigma^\delta$. Then $\mathscr{C}_\sigma^\delta \setminus \{(\zeta_0,w_0)\}$ is disconnected as a subset of $\mathbb{R} \times C_\sigma^{2+\beta}(\overline{\Omega})$, with exactly two connected components. One of these components is $\mathscr{C}_{\mathrm{loc}} \cap \{\delta \le \zeta < \zeta_0\}$.*

Proof. In what follows, $\mathscr{S}$ is always topologized as a subset of $\mathbb{R} \times C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$, while $\mathscr{S}_\sigma^\delta$ is topologized as a subset of $\mathbb{R} \times C_\sigma^{2+\beta}(\overline{\Omega})$. Assume for contradiction that $\mathscr{C}_\sigma^\delta \setminus \{(\zeta_0,w_0)\}$ is connected as a subset of $\mathscr{S}_\sigma^\delta$. By Lemma 5.4, it is connected as a subset of $\mathscr{S}$. By Lemma 5.3, $\mathscr{C}_{\mathrm{loc}} \cap \{\zeta < \zeta_0\}$ is a connected component of $\mathscr{S}$ which meets $\mathscr{C}_\sigma^\delta \setminus \{(\zeta_0,w_0)\}$, so this forces $\mathscr{C}_\sigma^\delta \setminus \{(\zeta_0,w_0)\}$ to be a subset of $\mathscr{C}_{\mathrm{loc}} \cap \{\zeta < \zeta_0\}$, a contradiction since $\mathscr{C}_{\mathrm{loc}} \cap \{\zeta > \zeta_0\} \subset \mathscr{C}_\sigma^\delta$. Thus $\mathscr{C}_\sigma^\delta \setminus \{(\zeta_0,w_0)\}$ has at least two connected components. Applying the implicit function theorem to $\mathscr{F}$ near (ζ_0,w_0), we conclude that $\mathscr{C}_\sigma^\delta \setminus \{(\zeta_0,w_0)\}$ has exactly two components. ◻

5.3. Topological degree. In this section we will summarize the Healey–Simpson degree [19], which we will use to prove Theorem 5.2 in section 5.4. We note that it might also be possible to use the C^2 degree of Fitzpatrick, Pejsachowicz, and Rabier [13]. First we define a notion of admissibility for linear maps, taken from Definition 4.7 and Remark 4.13 in [19].

DEFINITION 5.6. *Let X, Y_1, Y_2 be Banach spaces with X continuously embedded in Y_1, and set $Y = Y_1 \times Y_2$. We assume that X is endowed with weaker norms $\|\cdot\|_{X'}, \|\cdot\|_{X''}, \|\cdot\|_{X'''}$ such that*

$$C'''\|u\|_{X'''} \le C''\|u\|_{X''} \le C'\|u\|_{X'} \le \|u\|_X.$$

We also assume the existence of weaker norms $\|\cdot\|_{Y_2'}, \|\cdot\|_{Y_2''}, \|\cdot\|_{Y_2'''}$ for Y_2 and $\|\cdot\|_{Y_1'}$ for Y_1 satisfying similar inequalities. Letting X' denote X with the norm $\|\cdot\|_{X'}$ and so on, we consider a linear operator $L = (A, B)\colon X \to Y$ satisfying

$$(A, B) \in \mathscr{L}(X, Y) \cap \mathscr{L}(X', Y_1' \times Y_2'), \qquad B \in \mathscr{L}(X'', Y_2'') \cap \mathscr{L}(X''', Y_2'''), \tag{5.1}$$

where $\mathscr{L}(E, F)$ denotes the space of bounded linear operators $E \to F$. Such an operator L is said to be admissible if, in addition to (5.1)*, the following hold:*

(i) *L is a Fredholm operator of index* 0*.*
(ii) *B is surjective.*
(iii) *There exist constants $\beta, C_1, C_2 > 0$ such that*

$$\begin{aligned} C_1 &\left(\|u\|_X + |\kappa|\|u\|_{X'} + |\kappa|^{1/2}\|u\|_{X''} + |\kappa|^{\beta+1/2}\|u\|_{X'''} \right) \\ &\le \|(A - \kappa I)u\|_{Y_1} + |\kappa|^\beta \|(A - \kappa I)u\|_{Y_1'} \\ &\quad + \|Bu\|_{Y_2} + |\kappa|^\beta\|Bu\|_{Y_2'} + |\kappa|^{1/2}\|Bu\|_{Y_2''} + |\kappa|^{\beta+1/2}\|Bu\|_{Y_2'''} \end{aligned}$$

for all $u \in X$ and real $\kappa \ge C_2$.

(iv) *There exists an open neighborhood $\mathscr{N}$ of the ray $\{\mu : \mu \ge 0\} \subset \mathbb{C}$ such that $\Sigma(A, B) \cap \mathscr{N}$ consists of finitely many eigenvalues, each of finite algebraic multiplicity. Here, as in Definition* 3.8*, $\Sigma(A, B)$ is the spectrum of A, considered as an unbounded operator $\tilde{A}\colon X \to Y$ with domain $\mathscr{D}(\tilde{A}) = X \cap \ker B$.*

Using Definition 5.6, we next define admissibility for nonlinear operators. This is Definition 4.10 together with Remark 4.13 in [19].

DEFINITION 5.7. *In the setting of Definition* 5.6*, let $W \subset X$ be open and bounded and let $\overline{W}'$ denote $\overline{W}$ endowed with the X' topology and similarly for $\overline{W}''$, $\overline{W}'''$. A map $F = (F_1, F_2)\colon \overline{W} \to Y$ is* admissible *if the following hold:*

(i) *F and F_u have the regularity*

$$\begin{gathered} F \in C^2(W, Y) \cap C^0(\overline{W}, Y), \qquad F_u \in C^0(\overline{W}', \mathscr{L}(X', Y_1' \times Y_2')), \\ F_{2u} \in C^0(\overline{W}'', \mathscr{L}(X'', Y_2'')) \cap C^0(\overline{W}''', \mathscr{L}(X''', Y_1''')). \end{gathered}$$

(ii) *For each $u \in \overline{W}$, $F_u(u)$ is admissible according to Definition* 5.6*.*
(iii) *$F\colon \overline{W} \to Y$ is locally proper.*

Suppose that $F\colon \overline{W} \to Y$ is admissible and $y \in Y \setminus F(\partial W)$ is a regular value of F. By this we mean that $F_u(u)$ is surjective (and hence invertible since it is Fredholm of index 0) for all $u \in F^{-1}(y) \cap W$. Then $F^{-1}(y) \cap W$ is finite, and we define

$$\deg(F, W, y) = \sum_{u \in F^{-1}(y) \cap W} (-1)^{\nu(u)},$$

where $\nu(u)$ is the number, counted according to algebraic multiplicity, of positive eigenvalues in $\Sigma(F_u(u))$, which is finite by admissibility, and where the sum over the empty set is 0. If $y \notin F(\partial W)$ is not a regular value, we define $\deg(F, W, y)$ to be $\deg(F, W, \widetilde{y})$ for some nearby regular value $\widetilde{y}$ which exists by the Sard–Smale theorem; see [19].

We need two properties of the degree. The first is additivity.

LEMMA 5.8 (additivity). *Suppose that $W^1, W^2 \subset X$ are bounded open sets with $W^1 \cap W^2 = \varnothing$ and that $F\colon \overline{W}^1 \cup \overline{W}^2 \to Y$ is admissible. If $y \notin F(\partial W^1 \cup \partial W^2)$, then*

$$\deg(F, W^1 \cup W^2, y) = \deg(F, W^1, y) + \deg(F, W^2, y).$$

Proof. Let $\widetilde{y}$ be a regular value for $F|_{W^1 \cup W^2}$, close enough to y that

$$\deg(F, W^1 \cup W^2, y) = \deg(F, W^1 \cup W^2, \widetilde{y}), \qquad \deg(F, W^i, y) = \deg(F, W^i, \widetilde{y}),$$

for $i = 1, 2$. The statement then follows from

$$\sum_{u \in F^{-1}(\widetilde{y}) \cap (W^1 \cup W^2)} (-1)^{\nu(u)} = \sum_{u \in F^{-1}(\widetilde{y}) \cap W^1} (-1)^{\nu(u)} + \sum_{u \in F^{-1}(\widetilde{y}) \cap W^2} (-1)^{\nu(u)}. \qquad \square$$

The most important property of degree for us is invariance under homotopy, which is proven in Proposition 4.12 of [19] and the following remarks. For $\Upsilon \subset [0,1] \times W$ and $t \in [0,1]$, define the section

$$\Upsilon_t = \{u \in W : (t, u) \in \Upsilon\}. \tag{5.2}$$

DEFINITION 5.9. *For $\Upsilon \subset [0,1] \times W$ open, we say that $H\colon \overline{\Upsilon} \to Y$ is an* admissible generalized homotopy *if $H \in C^2(\Upsilon, Y)$ is proper and $H(t, \cdot\,)$ is admissible for each t. We call $t \in [0,1]$ the parameter of the homotopy.*

LEMMA 5.10 (homotopy invariance). *If $H\colon \overline{\Upsilon} \to Y$ is an admissible generalized homotopy, and $y \notin H(\partial \Upsilon_t)$ for $t \in [0,1]$, then*

$$\deg(H(0, \cdot\,), \Upsilon_0, y) = \deg(H(1, \cdot\,), \Upsilon_1, y).$$

5.4. Global continuation. In the notation of section 5.3, we take

$$\begin{aligned}
&X = \mathscr{X}^{\mathrm{e}}_\sigma, && \|\cdot\|_{X'} = |\sigma \cdot|_2, && \|\cdot\|_{X''} = |\sigma \cdot|_{1+\beta}, && \|\cdot\|_{X'''} = |\sigma \cdot|_1,\\
&Y_2 = C^{1+\beta}_{\sigma,\mathrm{e}}(T), && \|\cdot\|_{Y_2'} = |\sigma \cdot|_1, && \|\cdot\|_{Y_2''} = |\sigma \cdot|_\beta, && \|\cdot\|_{Y_2'''} = |\sigma \cdot|_0,\\
&Y_1 = C^{\beta}_{\sigma,\mathrm{e}}(\overline{\Omega}), && \|\cdot\|_{Y_1'} = |\sigma \cdot|_0,
\end{aligned}$$

where the spaces $\mathscr{X}^{\mathrm{e}}_\sigma, \mathscr{Y}^{\mathrm{e}}_\sigma$ are defined in (3.2) and (3.5). (We will not need to reference the spaces X, Y used in section 4 again.) The following lemma will allow us to apply Lemma 5.10 to our nonlinear operator $\mathscr{F}$.

LEMMA 5.11. *For any $\delta > 0$, $\mathscr{F}\colon [\delta, \alpha_{\mathrm{cr}} - \delta] \times \overline{\mathscr{U}_\sigma} \to \mathscr{Y}^{\mathrm{e}}_\sigma$ is an admissible generalized homotopy with parameter ζ. In particular, $\mathscr{F}|_{\overline{\Upsilon}}$ is an admissible generalized homotopy for any open $\Upsilon \subset (\delta, \alpha_{\mathrm{cr}} - \delta) \times \mathscr{U}_\sigma$.*

Proof. First we claim that for $(\zeta, w) \in [\delta, \alpha_{\mathrm{cr}} - \delta] \times \overline{\mathscr{U}_\sigma}$, the linear operator $(\mathscr{A}, \mathscr{B}) = \mathscr{F}_w(\zeta, w)$ is admissible according to Definition 5.6. Condition 1 is Corollary 3.5, condition 3 is a special case of Lemma 3.7, and condition 4 is Lemma 3.9. Finally, condition 2 is a consequence of condition 4: by 4, there exists $\kappa \in \mathbb{C}$ such that

$$(\mathscr{A} - \kappa I, \mathscr{B})\colon \mathscr{X}^{\mathrm{e}}_\sigma \to \mathscr{Y}^{\mathrm{e}}_\sigma = C^{2+\beta}_{\sigma,\mathrm{e}}(\overline{\Omega}) \times C^{1+\beta}_{\sigma,\mathrm{e}}(T)$$

is onto. Thus $\mathscr{B}\colon \mathscr{X}^{\mathrm{e}}_\sigma \to C^{1+\beta}_{\sigma,\mathrm{e}}(T)$ must be onto. Next we claim $\mathscr{F}(\zeta,\cdot)\colon \overline{\mathscr{U}_\sigma} \to \mathscr{Y}^{\mathrm{e}}_\sigma$ is admissible according to Definition 5.7 for $\delta < \zeta < \infty$. The regularity condition 1 is easily checked. We then have local properness 3 by Theorem 3.10. Finally, we use Theorem 3.10 yet again to get the local properness of $\mathscr{F}\colon (\delta, \alpha_{\mathrm{cr}} - \delta) \times \mathscr{U}_\sigma \to \mathscr{Y}^{\mathrm{e}}_\sigma$. Since $\mathscr{F}$ is C^2, we conclude that $\mathscr{F}$ is an admissible generalized homotopy. ∎

Proof of Theorem 5.2. We follow the proof of Theorem II.6.1 in [26]. By Lemma 5.5, we know that $\mathscr{C}^\delta_\sigma \setminus \{(\zeta_0, w_0)\}$ has two components, one of which is

$$\mathscr{C}^{\delta,-}_\sigma := \mathscr{C}_{\mathrm{loc}} \cap \{\delta \le \zeta < \zeta_0\}.$$

Assume for contradiction that the other component $\mathscr{C}^{\delta,+}_\sigma$ is bounded and does not meet the boundary of $(\delta, \alpha_{\mathrm{cr}} - \delta) \times \mathscr{U}_\sigma$. By local properness (Theorem 3.10), the closed set $\overline{\mathscr{C}^{\delta,+}_\sigma} = \mathscr{C}^{\delta,+}_\sigma \cup \{(\zeta_0, w_0)\}$ is compact.

Pick a point $(\zeta_1, w_1) \in \mathscr{C}_{\mathrm{loc}}$ with $\zeta_1 > \zeta_0$. By Theorem 4.1, the linear operator $\mathscr{F}_w(\zeta, w^\zeta)$ is invertible for each $\zeta_0 \le \zeta \le \zeta_*$. Therefore w^ζ is the unique solution of $\mathscr{F}(\zeta, w) = 0$ locally, that is, we can shrink ε_1 so that $w = w^\zeta$ whenever $\mathscr{F}(\zeta, w) = 0$, $\zeta_0 \le \zeta \le \zeta_1$, and $\|w - w^\zeta\|_{\mathscr{X}^{\mathrm{e}}_\sigma} \le \varepsilon_1$. Consider the open strip

$$R := \big\{(\zeta, w) \in \mathbb{R} \times \mathscr{X}^{\mathrm{e}}_\sigma : \zeta_0 < \zeta < \zeta_1,\ \|w - w^\zeta\|_{\mathscr{X}^{\mathrm{e}}_\sigma} < \varepsilon_1\big\},$$

as well as the portion of its boundary

$$\partial_w R := \big\{(\zeta, w) \in \mathbb{R} \times \mathscr{X}^{\mathrm{e}}_\sigma : \zeta_0 \le \zeta \le \zeta_1,\ \|w - w^\zeta\|_{\mathscr{X}^{\mathrm{e}}_\sigma} = \varepsilon_1\big\}.$$

We have just shown that $R \cap \mathscr{S} \subset \mathscr{C}_{\mathrm{loc}}$ and $\partial_w R \cap \mathscr{S} = \varnothing$. Defining sections $R_\zeta = \{w : (\zeta, w) \in R\}$ as in (5.2), the definition of the degree then gives

$$\deg(\mathscr{F}(\zeta,\cdot), R_\zeta, 0) = (-1)^{\nu(\zeta)} \ne 0, \qquad \zeta_0 < \zeta < \zeta_1, \tag{5.3}$$

where $\nu(\zeta)$ is the number of positive eigenvalues of $\mathscr{F}_w(\zeta, w^\zeta)$ counted according to multiplicity.

We now construct a bounded open neighborhood U of $\mathscr{C}^{\delta,+}_\sigma$ with the following properties:

(i) $R \subset U \subset (\delta, \alpha_{\mathrm{cr}} - \delta) \times \mathscr{U}_\sigma$,
(ii) $U \cap \partial_w R = \varnothing$,
(iii) $\mathscr{F} \ne 0$ on $\partial U \setminus \{(\zeta_0, w_0)\}$,
(iv) $(\zeta_0, w_0) \notin \partial(U \setminus R)$.

See Figure 3(b). By assumption, $\mathscr{C}^{\delta,+}_\sigma$ does not meet the boundary of $(\delta, \alpha_{\mathrm{cr}} - \delta) \times \mathscr{U}_\sigma$. Our above argument shows that $\mathscr{C}^{\delta,+}_\sigma$ does not meet $\partial_w R$. Since $\mathscr{C}^{\delta,+}_\sigma$ is a component of $\mathscr{C}^\delta_\sigma \setminus \{(\zeta_0, w_0)\}$, and $\mathscr{C}^\delta_\sigma$ is a component of $\mathscr{S}^\delta_\sigma$, $\mathscr{C}^{\delta,+}_\sigma$ cannot meet $\mathscr{S}^\delta_\sigma \setminus \mathscr{C}^{\delta,+}_\sigma$ either. Thus there exists $\varepsilon_2 > 0$ such that the distance between the compact set $\mathscr{C}^{\delta,+}_\sigma \setminus R$ and any of the closed sets

$$\partial((\delta, \alpha_{\mathrm{cr}} - \delta) \times \mathscr{U}_\sigma), \quad \partial_w R, \quad \mathscr{S}^\delta_\sigma \setminus \mathscr{C}^{\delta,+}_\sigma$$

is at least $2\varepsilon_2$. Let U^1 be the open ε_2-neighborhood of $\mathscr{C}^{\delta,+}_\sigma \setminus R$, and set $U = U^1 \cup R$. Properties (i) and (ii) are clear. Property (iii) follows from $\mathscr{F} \ne 0$ on $\partial U^1 \setminus R$ and $(\zeta_1, w_1) \in \mathscr{C}^{\delta,+}_\sigma \setminus R \subset U^1$. Finally, property (iv) holds because $U \setminus R \subset U^1$ is a positive distance away from $(\zeta_0, w_0) \in \mathscr{S}^\delta_\sigma \setminus \mathscr{C}^{\delta,+}_\sigma$.

Now we derive a contradiction by comparing the degree of $\mathscr{F}$ on various sections. By (iii), $\mathscr{F} \ne 0$ on $\partial U \setminus \{(\zeta_0, w_0)\}$. Thus homotopy invariance (Lemmas 5.10 and

5.11) implies that $\deg(\mathscr{F}(\zeta,\,\cdot\,),U_\zeta,0)$ is independent of ζ for $\zeta > \zeta_0$. Since $U_\zeta = \varnothing$ for ζ sufficiently close to α_{cr}, we get

$$\deg(\mathscr{F}(\zeta,\,\cdot\,),U_\zeta,0) = 0 \qquad \forall \zeta > \zeta_0.$$

Set $V = U \setminus \overline{R}$. By (iii) and (iv) we have $\mathscr{F} \neq 0$ on $\partial V \setminus \{(\zeta_1, w_1)\}$, so homotopy invariance implies that $\deg(\mathscr{F}(\zeta,\,\cdot\,),V_\zeta,0)$ is independent of ζ for $\zeta < \zeta_1$. Since $V_\zeta = \varnothing$ for ζ sufficiently close to 0, we have

$$\deg(\mathscr{F}(\zeta,\,\cdot\,),V_\zeta,0) = 0 \qquad \forall \zeta < \zeta_1.$$

From (ii) we have $U_\zeta = V_\zeta \cup R_\zeta$ and $V_\zeta \cap R_\zeta = \varnothing$ for $\zeta_0 < \zeta < \zeta_1$. Since $\mathscr{F} \neq 0$ on $\partial_w R$, the additivity of the degree (Lemma 5.8) gives

$$\deg(\mathscr{F}(\zeta,\,\cdot\,),V_\zeta,0) + \deg(\mathscr{F}(\zeta,\,\cdot\,),R_\zeta,0) = \deg(\mathscr{F}(\zeta,\,\cdot\,),U_\zeta,0) \qquad \forall \zeta_0 < \zeta < \zeta_1.$$

We've shown already that two of the above degrees are 0, leaving us with

$$\deg(\mathscr{F}(\zeta,\,\cdot\,),R_\zeta,0) = 0 \qquad \forall \zeta_0 < \zeta < \zeta_1,$$

which contradicts (5.3). ◻

5.5. Uniform regularity along the continuum. One of the possibilities in Theorem 5.2 is that $\mathscr{C}_\sigma^{\delta,+}$ is unbounded in $\mathbb{R} \times \mathscr{U}_\sigma$. Since we always have $0 < \zeta < \alpha_{\mathrm{cr}}$, this is equivalent to $|\sigma w|_{2+\beta}$ being unbounded along $\mathscr{C}_\sigma^{\delta,+}$. In this section we will show that, for supercritical solitary waves, $|\sigma w|_{2+\beta}$ is controlled by $|\sigma w|_0$ and $|w_s|_0$, while $|w|_{2+\beta}$ is controlled by $|w_s|_0$ alone. These estimates will allow us to establish uniform bounds along the continua $\mathscr{C}$ and $\mathscr{C}_\sigma^\delta$ in sections 5.6 and 5.7, addressing the possibility in Theorem 5.2 that $\mathscr{C}_\sigma^{\delta,+}$ is unbounded.

Proposition 5.12. *For each $K > 0$ there exists a constant C depending only on K such that all supercritical solitary waves (ζ, w) with $|w_s|_0 \le K$ satisfy $|w|_{2+\beta} \le C$.*

Proposition 5.13. *For each $K > 0$ there exists a constant C depending only on K and σ such that all supercritical solitary waves (ζ, w) with $|w_s|_0 + |\sigma w|_0 \le K$ satisfy $|\sigma w|_{2+\beta} \le C$.*

We will prove Propositions 5.12 and 5.13 in several steps:

1. Estimate $|w|_1$ in terms of $|w_s|_0$.
2. Estimate $|w|_{1+\beta'}$ in terms of $|w|_1$ for some $\beta' \in (0, \beta]$.
3. Estimate $|w|_{2+\beta'}$ in terms of $|w|_{1+\beta'}$.
4. Repeat step 3 with β' replaced by β.
5. Estimate $|\sigma w|_{2+\beta}$ in terms of $|\sigma w|_0$ and $|w|_{2+\beta}$.

Step 1 follows easily from Proposition 2.4.

Lemma 5.14. *Let (ζ, w) be a supercritical solitary wave. Then there exists a constant C depending only on γ so that $|w|_1 \le C(1 + |w_s|_0)$.*

Proof. By Proposition 2.4 we have $|w_x|_0 \le C(1 + |w_s|_0)$, and $|w|_0 \le |w_s|_0$ follows from writing $w(x,s) = \int_0^s w_s(x,t)\,dt$. ◻

To complete step 2, we use regularity results for two-dimensional nonlinear elliptic boundary problems. For convenience, set $y = (x, s)$. Fixing $R \in (0,1)$, we work on half-balls

$$B_R^- = \{y \in \mathbb{R}^2 : |y - (x_0, 1)| < R,\ s < 1\} \subset \Omega \tag{5.4}$$

centered at points $(x_0, 1) \in T$. Consider a nonlinear problem

$$F(y, D\varphi, D^2\varphi) = 0 \text{ in } B_R^-, \qquad G(\varphi, D\varphi) = 0 \text{ on } \partial B_R^- \cap T, \tag{5.5}$$

and assume that there exist positive constants c_1, c_2, c_3 such that

$$c_1 I \le F_r(x,p,r) \le c_2 c_1 I, \qquad |G_p(z,p)| \ge c_3 \tag{5.6}$$

for all $(x,z,p,r) \in B_R^- \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^2$. Here $\mathbb{S}^2$ is the space of real symmetric 2×2 matrices.

A simplified version of Theorem 1 in [32] then reads as follows.

THEOREM 5.15. *Fix $R \in (0,1)$ and a Hölder parameter $\beta \in (0,1)$, and let $F \in C^{0,1}(B_R^- \times \mathbb{R}^2 \times \mathbb{S}^2)$ and $G \in C^0(\mathbb{R}\times\mathbb{R}^2)$ satisfy (5.6) for some positive constants c_1, c_2, c_3. Suppose in addition that there exists a positive constant c_4 so that*

$$|F(y,p,0)| \le c_4, \qquad |F_p(y,p,r)| \le c_4(1+|r|), \qquad |F_y(y,p,r)| \le c_4$$

for all $(y,p,r) \in B_R^- \times \mathbb{R}^2 \times \mathbb{S}^2$, and

$$|G(z,p) - G(z',p')| \le c_4\big(|z-z'|^\beta + |p-p'|\big)$$

for all (z,p) and (z',p') in $\mathbb{R}\times\mathbb{R}^2$. Then for any $K>0$, there exist positive constants β' and C depending on $\beta, R, c_1, c_2, c_3, c_4$ so that any solution $\varphi \in C^{0,1}(B_R^-) \cap W^{3,2}_{\mathrm{loc}}(B_R^-)$ of (5.5) with $\sup(|\varphi| + |D\varphi|) \le K$ obeys

$$|\varphi|_{1+\beta'; B_{R/2}^-} \le C. \tag{5.7}$$

So that our formulas for F and G are simpler, we will apply Theorem 5.15 to equations (1.14a)–(1.14b) for $h = H+w$ and $\alpha = \alpha_{\mathrm{cr}} - \zeta$, instead of the corresponding equations (1.19a)–(1.19b) for w and ζ. Writing (1.14a)–(1.14b) in nondivergence form, we set

$$\begin{aligned} F(s,p,r) &= (1+p_1^2) r_{22} - 2p_1p_2 r_{12} + p_2^2 r_{11} + \gamma(-s) p_2^3, \\ G(z,p;\alpha) &= \frac{1+p_1^2}{2p_2^2} + \alpha(z-1) - \frac{\mu}{2}. \end{aligned}$$

We easily check that F and G satisfy the hypotheses of Theorem 5.15 when restricted to regions of the form $|z| + |p| \le K$ and $p_2 \ge \delta > 0$ with constants c_1, c_2, c_3, c_4 depending only on K, δ, β (and not on x_0). Modifying F and G using cutoff functions, we conclude that if the φ in Theorem 5.15 satisfies $\varphi_s \ge \delta > 0$, then the conclusion (5.7) holds with C and β' depending only on K, δ, β.

Using Theorem 5.15, we can now complete step 2.

LEMMA 5.16. *For each $K > 0$ there exists $C = C(K)$ and $\beta' = \beta'(K) \in (0,\beta]$ so that any supercritical solitary wave (ζ, w) with $|w_s|_0 < K$ satisfies $|w|_{1+\beta'} < C$.*

Proof. Let (ζ, w) solve (1.19), and for convenience set $h = H + w$. In what follows we use $C > 0$ and $\beta' \in (0,\beta]$ to denote constants depending only on K. By Lemma 5.14 we have $|w|_1 < C$, and hence $|h|_1 \le |H|_1 + |w|_1 < C$. By Proposition 2.4, we also have $\inf_\Omega h_s \ge \delta_*$, where $\delta_* > 0$ is independent of (ζ, w). Thus $\theta = h$ solves the uniformly elliptic equation

$$(1+h_x^2)\theta_{xx} - 2h_x h_s \theta_{xs} + h_s^2 \theta_{xx} = -\gamma h_s^3$$

in Ω, so by standard elliptic theory [16, Theorem 9.19] $h \in C^{3+\beta}(\Omega)$ and hence $h \in W^{3,2}_{\mathrm{loc}}(\Omega)$.

Now pick $x_0 \in \mathbb{R}$ and define $B_1^- = B_1^-(x_0)$ as in (5.4). Applying Theorem 5.15 to F and G on B_1^-, we see that h satisfies $|h|_{1+\beta'; B_{1/2}^-} \le C$, where β' and C depend only on K and not on the center x_0 of the half-ball B_1^-. We can also make a similar argument near the bottom boundary $s=0$ by setting $G(z,p)=p_1$. Applying these estimates to each half-ball centered at the boundary, we conclude $|h|_{1+\beta';\Omega'} \le C$, where $\Omega' = \mathbb{R} \times [(0,\frac{1}{4}) \cup (\frac{3}{4},1)]$. Combining this with a $C^{1+\beta'}$ interior estimate for quasilinear equations [16, Theorem 13.6], we have $|h|_{1+\beta'} < C$ and hence $|w|_{1+\beta'} < C$. ◻

To complete step 3, we differentiate $\mathscr{F}(\zeta,w)=0$ with respect to x and apply an (unweighted) Schauder-type estimate for divergence form equations.

LEMMA 5.17. *For each $K>0$ and $\beta' \in (0,\beta]$ there exists $C=C(K,\beta')$ so that any supercritical solitary wave (ζ,w) with $|w|_{1+\beta'} < K$ satisfies $|w|_{2+\beta'} < C$.*

Proof. Let (ζ,w) be a supercritical solitary wave with $|w|_{1+\beta'} < K$, and for convenience set $h = H+w$. Differentiating $\mathscr{F}(\zeta,w)=0$ with respect to x, we see that $\varphi = w_x$ is a weak ($C^{1+\beta'}$) solution to the divergence form elliptic equation $\mathscr{F}_{1w}(\zeta,w)\varphi = 0$,

$$
\begin{aligned}
\partial_s\left(\frac{1+w_x^2}{h_s^3}\partial_s\varphi - \frac{w_x}{h_s^2}\partial_x\varphi\right) + \partial_x\left(-\frac{w_x}{h_s^2}\partial_s\varphi + \frac{1}{h_s}\partial_x\varphi\right) &= 0 \qquad \text{in } \Omega,\\
\frac{1+w_x^2}{h_s^3}\partial_s\varphi + \frac{w_x}{h_s}\partial_x\varphi + (\alpha_{\mathrm{cr}}-\zeta)\varphi &= 0 \qquad \text{on } s=1
\end{aligned} \tag{5.8}
$$

with $\varphi = 0$ on $s=0$. Since $h_s \ge \delta_*$ by Proposition 2.4, the linear operator $\mathscr{F}_{1w}(\zeta,w)$ is uniformly elliptic. Moreover the coefficients in (5.8) have their $C^{\beta'}(\Omega)$ norms controlled by K, and the boundary operator $\mathscr{F}_{2w}(\zeta,w)$ is uniformly oblique. Thus the Schauder-type estimate Lemma A.2 gives $|w_x|_{1+\beta'} \le C|w_x|_0 \le C$. It remains to estimate $|w_{ss}|_{\beta'}$. Solving $\mathscr{F}_1(\zeta,w)=0$ for w_{ss} as in the proof of Lemma 4.6, we get

$$
w_{ss} = \frac{-h_s^2 w_{xx} + 2h_s w_x w_{xs} + \gamma H_s^3 w_x^2 - 3\gamma H_s^2 w_s - 3\gamma H_s w_s^2 - \gamma w_s^3}{1+w_x^2},
$$

and hence $|w|_{2+\beta'} < C$. ◻

We can now prove Proposition 5.12 by completing step 4.

Proof of Proposition 5.12. Let (ζ,w) be a supercritical solitary wave with $|w_s|_0 \le K$. In what follows we denote by $C>0$ constants depending only on K. By Lemma 5.16, we have $|w|_{1+\beta'} < C$, where $\beta' \in (0,\beta]$ depends only on K. Applying Lemma 5.17, we then have $|w|_{2+\beta'} < C$. In particular, this means $|w|_{1+\beta} < C$, so we can apply Lemma 5.17 again to get $|w|_{2+\beta} < C$. ◻

Finally, we complete step 5 by writing $\mathscr{F}(\zeta,w)=0$ in nondivergence form and applying the weighted Schauder estimate Lemma A.9.

Proof of Proposition 5.13. Let (ζ,w) be a supercritical solitary wave satisfying $|\sigma w|_0 < K$, and for convenience set $h = H+w$ and $\alpha = \alpha_{\mathrm{cr}} - \zeta$. By Proposition 5.12, we have $|w|_{2+\beta} < C(K)$.

Writing $\mathscr{F}(\zeta,w)=0$ in nondivergence form, we see that $\varphi = w$ solves

$$
\begin{aligned}
(1+w_x^2)\varphi_{ss} - 2h_s h_x \varphi_{xs} + h_s^2\varphi_{xx} + b_1\varphi_x + b_2\varphi_s &= 0 \quad \text{in } \Omega,\\
w_x\varphi_x - (\mu w_s + 2\sqrt{\mu})\varphi_s + c\varphi &= 0 \quad \text{on } s=1,
\end{aligned} \tag{5.9}
$$

together with $\varphi = 0$ on $s=0$, where

$$
b_1 = -\gamma H_s^3 w_x, \qquad b_2 = 3\gamma H_s^2 + 3\gamma H_s w_s + \gamma w_s^2, \qquad c = \frac{4\alpha}{\sqrt{\mu}} w_s + 2\alpha w_s^2 + \frac{2\alpha}{\mu}.
$$

Since $h_s \geq \delta_*$ by Proposition 2.4, (5.9) is a uniformly elliptic equation for φ. On $s = 1$ we have $0 \leq H_s + w_s = 1/\sqrt{\mu} + w_s$, so the coefficient of φ_s in the second line of (5.9) satisfies

$$\mu w_s + 2\sqrt{\mu} \geq -\mu/\sqrt{\mu} + 2\sqrt{\mu} = \sqrt{\mu}.$$

Thus the boundary condition in (5.9) is uniformly oblique. Moreover, we can bound the $C^{1+\beta}(\Omega)$ norms of the coefficients appearing in (5.9) in terms of $|w|_{2+\beta} < K$. Using the weighted Schauder estimate Lemma A.9, we conclude that

$$|\sigma w|_{2+\beta} \leq C_1(K, \sigma)|\sigma w|_0 \leq C_2(K, \sigma). \qquad \square$$

5.6. Fixing the weight function. This section is devoted to the proof of Proposition 5.18, which asserts that if a collection $\mathscr{W}$ of supercritical waves has $|w_s|_0$ uniformly bounded, then there exists a weight function σ so that $|\sigma w|_{2+\beta}$ is also uniformly bounded along $\mathscr{W}$. This will allow us to fix σ in section 5.7 and avoid an alternative in Theorem 1.1 involving the weight function. We will prove Proposition 5.18 by combining the equidecay result Proposition 2.5, the uniform bounds in Propositions 5.12 and 5.13, and an elementary fact about monotone functions.

Proposition 5.18. *Let $\mathscr{W}$ be a family of supercritical solitary waves with $\sup_{(\zeta,w)\in\mathscr{W}} |w_s|_0 < \infty$. Then there exists a strictly positive even function $\sigma \in C^\infty(\mathbb{R})$ satisfying* (3.1) *so that*

$$\sup_{(\zeta,w)\in\mathscr{W}} |\sigma(x)w|_{2+\beta} < \infty.$$

Proof. By Proposition 5.12, we have $\sup_{(\zeta,w)\in\mathscr{W}} |w|_{2+\beta} < \infty$. Thus by Proposition 2.5, $\mathscr{W}$ has the equidecay property

$$\lim_{x\to\pm\infty} \sup_{(\zeta,w)\in\mathscr{W}} \sup_{s\in[0,1]} |w(x,s)| = 0.$$

In particular, the function

$$F(x) := \sup_{(\zeta,w)\in\mathscr{W}} \sup_{s\in[0,1]} |w(x,s)|$$

is even with $F(x) \to 0$ as $x \to \pm\infty$. By the monotonicity and elevation of supercritical solitary waves, Propositions 2.1 and 2.2, f is monotone decreasing for $x > 0$, and $F(x) > 0$ for all $x \in \mathbb{R}$.

Our candidate weight function is $1/F(x)$, which is even, positive, goes to ∞ as $x \to \pm\infty$, and is monotone increasing for $x > 0$. Assume for the moment that we can find a smooth function $\sigma \leq 1/F$ with the above properties and

$$\lim_{x\to\pm\infty} \frac{D^k\sigma}{\sigma} = 0 \quad \text{for } k \geq 1.$$

Then σ is a weight function satisfying the hypotheses of section 3.1, in particular condition (3.1), as well as

$$\sup_{(\zeta,w)\in\mathscr{W}} |\sigma w|_0 \leq \sup_{(\zeta,w)\in\mathscr{W}} |w/F(x)|_0 \leq 1.$$

Applying Proposition 5.13, we have $\sup_{(\zeta,w)\in\mathscr{W}} |\sigma w|_{2+\beta} < \infty$ as desired. Thus the proof is complete, provided we can construct an appropriate σ in terms of $1/F$, which is the content of Lemma 5.19 below. $\square$

LEMMA 5.19. *Let $f\colon [0,\infty) \to (0,\infty)$ be a monotone increasing function with $f(x) \to \infty$ as $x \to \infty$. Then there exists a smooth, monotone increasing function $g\colon [0,\infty) \to (0,\infty)$ with $g \le f$ and $g(x) \to \infty$, such that*

$$\lim_{x\to\infty} \frac{D^k g}{g} \to 0, \qquad k = 1,2,3,\dots. \tag{5.10}$$

Proof. Define a sequence a_n inductively by

$$a_0 = f(0), \qquad a_{n+1} = \min\left\{f(n+1),\ (1+\tfrac{1}{n})a_n\right\}.$$

We claim that a_n has the properties

$$a_n > 0, \qquad a_n \le f(n), \qquad a_n \le a_{n+1}, \qquad a_n \to \infty, \qquad \frac{a_{n+1}}{a_n} \to 1.$$

Now $a_n > 0$ and $a_n \le f(n)$ are clear from the definition. Combining this with the monotonicity of f, we easily check that $a_n \le a_{n+1}$. If $a_{n+1} = f(n+1)$ infinitely often, then the monotonicity of a_n and $f(x) \to \infty$ imply $a_n \to \infty$. On the other hand, if $a_{n+1} = (1+1/n)a_n$ for $n \ge N$, then $\prod_{n=1}^{\infty}(1+1/n) = \infty$ also implies $a_n \to \infty$. Finally, $a_n \le a_{n+1} \le (1+1/n)a_n$ forces $a_{n+1}/a_n \to 1$.

Now we construct g in terms of a_n. Let $\varphi\colon [0,1] \to [0,1]$ be a smooth, monotone increasing function with $\varphi(x) = 0$ for $x < 1/4$ and $\varphi(x) = 1$ for $x > 3/4$. Setting $a_{-1} = a_0$, we define g piecewise by

$$g(x) = a_{n-1} + (a_n - a_{n-1})\varphi(x-n), \qquad x \in [n, n+1].$$

We easily check that g is smooth and monotone increasing, and also that

$$a_{n-1} \le g(x) \le a_n \le f(x), \qquad x \in [n, n+1].$$

In particular, since $a_n \to \infty$, we have $g \to \infty$ as $x \to \infty$. Taking derivatives, we find

$$D^k g(x) = (a_n - a_{n-1}) D^k \varphi(x-n), \qquad x \in [n, n+1],$$

and hence

$$\left|\frac{D^k g(x)}{g(x)}\right| \le \frac{a_n - a_{n-1}}{a_n}\|D^k\varphi\|_{L^\infty} = \left(1 - \frac{a_{n-1}}{a_n}\right)\|D^k\varphi\|_{L^\infty}, \qquad x \in [n, n+1].$$

Sending $n \to \infty$ and using $a_{n-1}/a_n \to 1$, we obtain (5.10) as desired. ∎

5.7. Proof of the main theorem. We are now in a position to prove our main result.

Proof of Theorem 1.3. Let $(\zeta, w) \in \mathscr{C}$. By Lemma 5.3, w is nontrivial, $w \not\equiv 0$. Thus by Proposition 2.1, we have the elevation condition $w(x,1) > 0$ for $x \in \mathbb{R}$. Applying Proposition 2.2 we get the monotonicity condition $w_x < 0$ for $x > 0$ and $0 < s \le 1$.

First assume alternative (ii) holds, i.e., there exists a sequence $(\zeta_n, w_n) \in \mathscr{C}$ with $\zeta_n \nearrow \alpha_{\mathrm{cr}}$. We claim that we can extract a subsequence with $\lim_{n\to\infty} w_n(0,1) \ge d^*/d - 1$. If not, then

$$\sup_n w_n(0,1) = M < d^*/d - 1. \tag{5.11}$$

But then Proposition 2.3 gives $\alpha_{\mathrm{cr}} - \zeta_n > C > 0$ for all n, a contradiction.

From now on we assume alternative (i) does not hold, i.e., $\sup_{(\zeta,w)\in\mathscr{C}}|w_s|_0 < \infty$. Applying Proposition 5.18, there exists a smooth even weight function σ satisfying (3.1) with

$$\sup_{(\zeta,w)\in\mathscr{C}} |\sigma w|_{2+\beta} < \infty. \tag{5.12}$$

Let $\delta \in (0,\zeta_*)$. We claim that $\mathscr{C}^\delta_\sigma \setminus \mathscr{C}_{\mathrm{loc}}$ contains a solution with either $\zeta = \delta$ or $\zeta = \alpha_{\mathrm{cr}} - \delta$. To see this, we first pick $(\zeta_0, w_0) \in \mathscr{C}_{\mathrm{loc}} \cap \mathscr{C}^\delta_\sigma$. By Theorem 5.2, $\mathscr{C}^\delta_\sigma \setminus \{(\zeta_0,w_0)\}$ has exactly two connected components, $\mathscr{C}_{\mathrm{loc}} \cap \{\delta \le \zeta < \zeta_0\}$ and another component $\mathscr{C}^{\delta,+}_\sigma$ which is either unbounded or meets the boundary of $(\delta, \alpha_{\mathrm{cr}}-\delta)\times\mathscr{U}_\sigma$. Since $0 < \zeta < \alpha_{\mathrm{cr}}$ is always bounded and

$$\sup_{(\zeta,w)\in\mathscr{C}^{\delta,+}_\sigma} |\sigma w|_{2+\beta} \le \sup_{(\zeta,w)\in\mathscr{C}} |\sigma w|_{2+\beta} < \infty$$

by (5.12) and $\mathscr{C}^{\delta,+}_\sigma \subset \mathscr{C}$ (Lemma 5.4), we conclude that $\mathscr{C}^{\delta,+}_\sigma$ meets the boundary of $(\delta, \alpha_{\mathrm{cr}} - \delta) \times \mathscr{U}_\sigma$. By Proposition 2.4, all $(\zeta, w) \in \mathscr{C}$ have $h_s \ge \delta_*$ and hence $w \in \mathscr{U}_\sigma$. Therefore $\mathscr{C}^{\delta,+}_\sigma$ cannot meet $[\delta, \alpha_{\mathrm{cr}} - \delta] \times \partial\mathscr{U}_\sigma$ and must instead meet $\{\delta, \alpha_{\mathrm{cr}}-\delta\}\times\mathscr{U}_\sigma$. Since $\mathscr{C}_{\mathrm{loc}}$ only contains solutions with $\zeta < \zeta_* < \alpha_{\mathrm{cr}}-\delta$, any solution $(\alpha_{\mathrm{cr}} - \delta, w) \in \mathscr{C}^{\delta,+}_\sigma$ will not lie on $\mathscr{C}_{\mathrm{loc}}$. Similarly, since $\mathscr{C}^{\delta,+}_\sigma$ and $\mathscr{C}_{\mathrm{loc}} \cap \{\delta \le \zeta < \zeta_0\}$ are disjoint, any solution $(\delta, w) \in \mathscr{C}^{\delta,+}_\sigma$ will not lie on $\mathscr{C}_{\mathrm{loc}}$. This proves the claim.

Sending $\delta = 1/n \to 0$, we have proved the existence of a sequence (ζ_n, w_n) in $\mathscr{C} \setminus \mathscr{C}_{\mathrm{loc}}$ with either $\zeta_n \searrow 0$ or $\zeta_n \nearrow \alpha_{\mathrm{cr}}$. The second possibility is alternative (ii), which we have already dealt with, so assume that $\zeta_n \to 0$. Because of (5.12), $|\sigma w_n|_{2+\beta}$ is uniformly bounded, so we can extract a subsequence so that $|\sqrt{\sigma}(w_n - w)|_2 \to 0$ for some $w \in C^{2+\beta}_\sigma(\overline{\Omega})$. We easily check that $(0, w)$ satisfies (1.19) as well as the weak monotonicity condition $w_x \ge 0$ for $x > 0$. If $w \equiv 0$, then $|w_n|_2 \to 0$ as $n \to \infty$, so by part (iii) of Theorem 4.1, (ζ_n, w_n) lies on $\mathscr{C}_{\mathrm{loc}}$ for n sufficiently large. But this contradicts $(\zeta_n, w_n) \in \mathscr{C} \setminus \mathscr{C}_{\mathrm{loc}}$. Thus $w \not\equiv 0$, so Proposition 2.1 implies the elevation condition $w(x, 1) > 0$ for $x \in \mathbb{R}$, which is alternative (iii). ∎

This completes the proof of Theorem 1.3, and hence, by Proposition 1.4, of Theorem 1.1.

Appendix. Elliptic problems in infinite strips. In this appendix we will prove results about elliptic problems in unbounded domains $\Omega = \mathbb{R}^n \times (0, 1)$ which are needed in sections 3 and 5. The main difficulty is the unboundedness of domain; in a bounded domain most of this appendix would either be unnecessary or would follow directly from standard elliptic theory. Because of this loss of compactness, the usual proofs of local properness using Schauder estimates no longer work. Recall from section 3 that we call a nonlinear mapping $F : X \to Y$ locally proper if $F^{-1}(K) \cap D$ is compact whenever $K \subset Y$ is compact and $D \subset X$ is closed and bounded. We will prove local properness using ideas from Volpert and Volpert [48], who consider general elliptic systems in quite general unbounded domains. Our setting is much simpler, and we will provide much more direct proofs of results in [48] for the reader's convenience.

In Appendix A.1, we will use translation invariance to prove a very mild extension of the usual Schauder estimate, and also state a Schauder-type estimate for divergence form equations from [10]. In Appendix A.2, we will specialize to equations with a particular divergence structure, and prove a sufficient condition for invertibility. We will begin following [48] in Appendix A.3, where we will prove local properness for elliptic operators. The proof involves the so-called limiting operators obtained by sending $|x| \to \infty$ in the coefficients. We will introduce weighted Hölder spaces in Appendix A.4, and prove weighted versions of the lemmas from Appendices A.1–A.3. Most of these weighted lemmas will require a subexponential growth assumption (A.11) on the weight function σ, but Schauder estimates will only require a weaker condition (A.12). In Appendix A.5, we will prove local properness for nonlinear elliptic operators in weighted Hölder spaces. The proof will use the weight to control quadratic terms, essentially reducing the problem to the linear one treated in Appendix A.3. Finally, in Appendix A.6, we will extend the above results to functions and operators with a reflection symmetry.

A.1. Schauder estimates. Set $\Omega = \mathbb{R}^n \times (0,1)$ and let $\Gamma_1 = \mathbb{R}^n \times \{1\}$ and $\Gamma_0 = \mathbb{R}^n \times \{0\}$ be the upper and lower boundaries of Ω. In this section we're interested in elliptic boundary value problems of the form

$$Au = f \quad \text{in } \Omega, \qquad Bu = g \quad \text{on } \Gamma_1, \qquad u = 0 \quad \text{on } \Gamma_0, \tag{A.1}$$

where

$$Au = a^{ij} D_{ij} u + b^i D_i u + cu, \qquad Bu = \gamma^i D_i u + \alpha u. \tag{A.2}$$

Fixing $\beta \in (0,1)$, we assume the regularity $a^{ij}, b^i, c \in C_{\mathrm{b}}^{\beta}(\overline{\Omega})$ and $\alpha, \gamma^i \in C_{\mathrm{b}}^{1+\beta}(\Gamma_1)$. We also assume that A is uniformly elliptic and B is uniformly oblique, that is,

$$a^{ij} = a^{ji}, \qquad a^{ij}\xi_i\xi_j \geq c|\xi|^2, \qquad |\gamma^{n+1}| \geq c, \tag{A.3}$$

for some positive constant c. We are primarily interested in two-dimensional strips with $n = 1$, but take $n = 2$ in the proof of Lemma 3.7. By standard elliptic theory [1], solutions $u \in C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$ of (A.1) satisfy a Schauder estimate

$$|u|_{2+\beta} \leq C(|f|_\beta + |g|_{1+\beta} + |u|_0), \tag{A.4}$$

where the constant C depends only on the ellipticity and obliqueness constants and the stated norms of the coefficients. In fact, the requirement $u \in C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$ can be weakened to $u \in C_{\mathrm{b}}^{0}(\overline{\Omega}) \cap C^{2+\beta}(\overline{\Omega})$.

Lemma A.1. *Suppose that $u \in C_{\mathrm{b}}^{0}(\overline{\Omega}) \cap C^{2+\beta}(\overline{\Omega})$ satisfies* (A.1) *with $f \in C_{\mathrm{b}}^{\beta}(\overline{\Omega})$ and $g \in C_{\mathrm{b}}^{1+\beta}(\overline{\Omega})$. Then $u \in C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$. In particular, u satisfies the Schauder estimate* (A.4).

Proof. For simplicity, we only give the proof for $n = 1$. Let $x_0 \in \mathbb{R}$, and consider a rectangle $R = (x_0 - 1, x_0 + 1) \times (0,1)$. We let $2R = (x_0 - 2, x_0 + 2) \times (0,1)$ be the corresponding rectangle with twice the width. Combining Lemmas 6.4 and 6.29 from [16], we see that

$$|u|_{2+\beta;R} \leq C\big(|f|_{\beta;2R} + |g|_{k+\beta;\Gamma_1 \cap \overline{2R}} + |u|_{0;2R}\big), \tag{A.5}$$

where the constant C does not depend on x_0. Since $|u|_{2+\beta} \leq C \sup_R |u|_{2+\beta;R}$, we can take the supremum of both sides of (A.5) over R and recover (A.4). $\square$

For divergence form equations we also have Schauder-type estimates which demand less regularity on the coefficients. Consider the problem

$$D_i(a^{ij} D_j u) = 0 \quad \text{in } \Omega, \qquad \gamma^i D_i u + \alpha u = 0 \quad \text{on } \Gamma_1, \qquad u = 0 \quad \text{on } \Gamma_0, \tag{A.6}$$

where a^{ij}, γ^i satisfy the ellipticity and obliqueness condition (A.3).

LEMMA A.2. *Suppose that $a^{ij}, \gamma^i, \alpha \in C^\beta_{\mathrm{b}}(\overline{\Omega})$ and that $u \in C^1_{\mathrm{b}}(\overline{\Omega})$ is a weak solution of* (A.6). *Then u has the additional regularity $u \in C^{1+\beta}_{\mathrm{b}}(\overline{\Omega})$. Moreover $|u|_{1+\beta} \leq C|u|_0$, where the constant C depends only on the dimension, the stated norms of the coefficients, and the ellipticity and obliqueness constants.*

Proof. In a periodic strip this follows from Theorem 3 in [9]. Since this theorem is based on local estimates, we can extend it to the infinite strip as in the proof of Lemma A.1. ☐

A.2. Invertibility for divergence form equations. We now set $\Omega = \mathbb{R} \times (0,1)$, and write points in Ω as (x,y). We also specialize to equations with the special divergence form structure

$$D_i(a^{ij}D_ju) = f \text{ in } \Omega, \qquad -a^{2j}D_ju + \alpha u = g \text{ on } \Gamma_1, \qquad u = 0 \text{ on } \Gamma_0, \tag{A.7}$$

where $\alpha \in \mathbb{R}$ is a parameter and $a^{ij} \in C^{1+\beta}_{\mathrm{b}}(\overline{\Omega})$. Note that the uniform ellipticity of a^{ij} implies the uniform obliqueness of the boundary operator on Γ_1. Letting H be the Hilbert space

$$H = \{u \in H^1(\Omega) : u|_{\Gamma_0} \equiv 0 \text{ in the trace sense}\},$$

we call $u \in H$ a weak solution of (A.7) if

$$\iint_\Omega a^{ij}D_iuD_j\varphi\,dx\,dy - \alpha\int_{\Gamma_1} u\varphi\,dx = -\iint_\Omega f\varphi\,dx + \int_{\Gamma_1} g\varphi\,dx$$

for all $\varphi \in H$. By the usual Lax–Milgram arguments, (A.7) will have a unique weak solution for any $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_1)$ provided that the associated bilinear form is coercive. For $\alpha < 0$ this follows from $\|u\|_H \leq C\|Du\|_{L^2}$, which holds for functions $u \in H$ since they vanish on Γ_0. So assume $\alpha \geq 0$ and let $u \in H$ be smooth. Then

$$|u(x,1)|^2 = \left|\int_0^1 D_2u(x,y)\,dy\right|^2 \leq \left(\int_0^1 \frac{a^{11}}{\det(a^{ij})}\,dy\right)\left(\int_0^1 \frac{\det(a^{ij})}{a^{11}}|D_2u|^2\,dy\right)$$

for each $x \in \mathbb{R}$. Assuming that

$$M := \sup_{x\in\mathbb{R}} \int_0^1 \frac{a^{11}}{\det(a^{ij})}\,dy < \frac{1}{\alpha}, \tag{A.8}$$

we then easily check that

$$\begin{aligned}\iint_\Omega a^{ij}D_iuD_ju\,dx\,dy - \alpha\int_{\Gamma_1} u^2\,dx &\geq \iint_\Omega \left(a^{ij}D_iuD_ju - \alpha M\frac{\det(a^{ij})}{a^{11}}(D_2u)^2\right)dx\,dy \\ &\geq C\|u\|_H^2.\end{aligned}$$

Thus we have proved the following lemma.

LEMMA A.3. *If* (A.8) *holds, then* (A.7) *has a unique weak solution $u \in H$ whenever $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_1)$.*

By a perturbation argument, we easily obtain the following.

COROLLARY A.4. *Fix α satisfying* (A.8). *Then there exists $\varepsilon > 0$ so that the problem*

$$D_i(a^{ij}D_ju) + b^iD_iu + cu = f \text{ in } \Omega, \quad -a^{2j}D_ju + (\alpha+\gamma)u = g \text{ on } \Gamma_1,\ u = 0 \text{ on } \Gamma_0$$

has a unique weak solution $u \in H$ whenever $f \in L^2(\Omega)$, $g \in L^2(\Gamma_1)$, and $|b^i|_0$, $|c|_0$, and $|\gamma|_1$ are all less than ε.

Using Corollary A.4, we can then prove invertibility in Hölder spaces.

LEMMA A.5. *If α satisfies* (A.8)*, then* (A.7) *has a unique solution $u \in C^{2+\beta}_{\mathrm{b}}(\overline{\Omega})$ for each $f \in C^{\beta}_{\mathrm{b}}(\overline{\Omega})$ and $g \in C^{1+\beta}_{\mathrm{b}}(\Gamma_1)$.*

Proof. Fix α satisfying (A.8), and for $\varepsilon > 0$ define $\rho_\varepsilon(x) = \operatorname{sech} \varepsilon x$, $u_\varepsilon = \rho_\varepsilon u$, $f_\varepsilon = \rho_\varepsilon f$, and $g_\varepsilon = \rho_\varepsilon g$. Then (A.7) is equivalent to

$$D_i(a^{ij} D_j u_\varepsilon) + b^i_\varepsilon D_i u_\varepsilon + c_\varepsilon u_\varepsilon = f_\varepsilon \text{ in } \Omega, \quad -a^{2j} D_j u_\varepsilon + (\alpha_\varepsilon + \alpha) u_\varepsilon = g_\varepsilon \text{ on } \Gamma_1, \tag{A.9}$$

together with $u_\varepsilon = 0$ on Γ_0, where

$$b^i_\varepsilon = -\frac{2D_i\rho_\varepsilon}{\rho_\varepsilon} a^{ij}, \quad c_\varepsilon = -\frac{D_{ij}\rho_\varepsilon}{\rho_\varepsilon} a^{ij} + 2\frac{D_i\rho_\varepsilon D_j\rho_\varepsilon}{\rho_\varepsilon^2} a^{ij} + \frac{D_j\rho_\varepsilon}{\rho_\varepsilon} D_i a^{ij}, \quad \alpha_\varepsilon = \frac{D_j\rho_\varepsilon}{\rho_\varepsilon} a^{2j}.$$

We observe that $|b^i_\varepsilon|_\beta, |c_\varepsilon|_\beta, |\alpha_\varepsilon|_{1+\beta} \to 0$ as $\varepsilon \to 0$. Picking $0 < \varepsilon < \varepsilon_0$ with ε_0 sufficiently small, Corollary A.4 implies that (A.9) has a unique solution $u_\varepsilon \in H$ whenever $f_\varepsilon \in L^2(\Omega)$ and $g_\varepsilon \in L^2(\Gamma_1)$. Since $u \in C^{2+\beta}_{\mathrm{b}}(\overline{\Omega})$ implies $u_\varepsilon = \rho_\varepsilon u \in H$, we have in particular that solutions $u \in C^{2+\beta}_{\mathrm{b}}(\overline{\Omega})$ of (A.7) are unique. It remains to show existence. Fix $f \in C^{\beta}_{\mathrm{b}}(\overline{\Omega})$, $g \in C^{1+\beta}_{\mathrm{b}}(\Gamma_1)$, and note that $f_\varepsilon \in L^2(\Omega)$ and $g_\varepsilon \in L^2(\Gamma_1)$ for any $\varepsilon > 0$. Therefore there exists a unique weak solution $u_\varepsilon \in H$ of (A.9) for each $0 < \varepsilon < \varepsilon_0$. By standard elliptic theory (Theorems 8.8 and 9.19 in [16]), $u_\varepsilon \in C^{2+\beta}(\overline{\Omega}) \cap C^{\beta}_{\mathrm{b}}(\overline{\Omega})$ solves (A.7). Our Schauder estimate Lemma A.1 and uniqueness then give $u_\varepsilon \in C^{2+\beta}_{\mathrm{b}}(\overline{\Omega})$ with

$$|u_\varepsilon|_{2+\beta} \le C(|f_\varepsilon|_\beta + |g_\varepsilon|_{1+\beta}) \le C(|f|_\beta + |g|_{1+\beta}),$$

where the constant C is independent of $\varepsilon \in (0, \varepsilon_0)$. In the second inequality we've used the fact that $|\rho_\varepsilon|_{1+\beta}$ is uniformly bounded as $\varepsilon \to 0$. Thus $|u_\varepsilon|_{2+\beta}$ is bounded uniformly in ε, and we can take a subsequence $\varepsilon_n \to 0$ so that $u_{\varepsilon_n} \to u$ in $C^2_{\mathrm{loc}}(\overline{\Omega})$ for some $u \in C^{2+\beta}_{\mathrm{b}}(\overline{\Omega})$. Since $f_\varepsilon \to f$ in $C^0_{\mathrm{loc}}(\overline{\Omega})$, $g_\varepsilon \to g$ in $C^1_{\mathrm{loc}}(\Gamma_1)$, and $|b^i_\varepsilon|_\beta, |c_\varepsilon|_\beta, |\alpha_\varepsilon|_{1+\beta} \to 0$, we conclude that u solves (A.7). ∎

We note that the condition (A.8) appearing in Lemma A.5 is sharp in the following sense. Suppose that a^{ij} is diagonal and depends only on the vertical variable y. Then $u(y) = \int_0^y \frac{dy'}{a^{22}}$ has $u(0) = 0$ and $D_i(a^{ij} D_j u) = 0$ in Ω, with $a^{2j} D_j u + \alpha u = 0$ if and only if equality holds in (A.8).

Allowing for complex-valued functions, we obtain a similar result for an eigenvalue problem.

LEMMA A.6. *Fix α satisfying* (A.8)*. Then there exists $\kappa_0 < 0$ so that, for any $\kappa \in \mathbb{C} \setminus (-\infty, \kappa_0]$ and complex-valued $f \in C^{\beta}_{\mathrm{b}}(\overline{\Omega})$, $g \in C^{1+\beta}_{\mathrm{b}}(\Gamma_1)$, the problem*

$$D_i(a^{ij} D_j u) - \kappa u = f \text{ in } \Omega, \qquad -a^{2j} D_j u + \alpha u = g \text{ on } \Gamma_1, \qquad u = 0 \text{ on } \Gamma_0$$

has a unique (complex-valued) solution $u \in C^{2+\beta}_{\mathrm{b}}(\overline{\Omega})$.

A.3. Limiting problems and properness. We continue to set $\Omega = \mathbb{R} \times (0,1)$ and to write points in Ω as (x, y). Define the Banach spaces

$$X_{\mathrm{b}} = \{u \in C^{2+\beta}_{\mathrm{b}}(\overline{\Omega}) : u|_{\Gamma_0} \equiv 0\}, \qquad Y_{\mathrm{b}} = C^{\beta}_{\mathrm{b}}(\overline{\Omega}) \times C^{k+\beta}_{\mathrm{b}}(\Gamma_1).$$

Letting A, B be as in (A.2), we can think of (A, B) as a bounded linear operator $L = (A, B)\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$. In this section we will give sufficient conditions for L to be locally proper. For linear operators, local properness is equivalent to being semi-Fredholm with index $< +\infty$, i.e., having a closed range and finite-dimensional kernel.

Suppose that, as $x \to \pm\infty$,

$$a^{ij}(x,y) \to \widetilde{a}^{ij}(y), \quad b^i(x,y) \to \widetilde{b}^i(y), \quad c(x,y) \to \widetilde{c}(y), \quad \alpha(x) \to \widetilde{\alpha}, \quad \gamma^i(x) \to \widetilde{\gamma}^i,$$

where $\widetilde{a}^{ij}, \widetilde{b}^i, \widetilde{c} \in C^\beta[0,1]$. For any sequence x_n with $|x_n| \to \infty$ we can define shifted coefficients $a_n^{ij}(x,y) = a^{ij}(x + x_n, y)$. Since the a_n^{ij} are uniformly bounded in $C_{\mathrm{b}}^\beta(\overline{\Omega})$, we can extract a subsequence so that $a_n^{ij} \to \widetilde{a}^{ij}$ in $C_{\mathrm{loc}}^0(\overline{\Omega})$. Analogous statements hold for the other shifted coefficients b_n^i, c_n. Performing these extractions, we define shifted and limiting operators

$$\begin{aligned} A_n u &= a_n^{ij} D_{ij} u + b_n^i D_i u + c_n u, &\quad B_n u &= \alpha_n u + \beta_n^i D_i u, &\quad L_n &= (A_n, B_n), \\ \widetilde{A} u &= \widetilde{a}^{ij} D_{ij} u + \widetilde{b}^i D_i u + \widetilde{c} u, &\quad \widetilde{B} u &= \widetilde{\alpha} u + \widetilde{\gamma}^i D_i u, &\quad \widetilde{L} &= (\widetilde{A}, \widetilde{B}). \end{aligned}$$

LEMMA A.7. *Assume the homogeneous limiting problem $\widetilde{L}u = 0$ has no nontrivial solutions $u \not\equiv 0$ in X_{b}. Then $L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is locally proper.*

Proof. Let u_n be a bounded sequence in X_{b} such that $Lu_n \to f = (f_1, f_2)$ in Y_{b}. We need to show that u_n has a subsequence converging in X_{b}. Extracting a subsequence, $u_n \to u$ in $C_{\mathrm{loc}}^2(\overline{\Omega})$ with $u \in C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$ and $Lu = f$. By the Schauder estimate (A.4), it suffices to show $u_n \to u$ in $C_{\mathrm{b}}^0(\overline{\Omega})$. Assume that this is not true. Since $u_n \to u$ locally, we can extract a subsequence so that

$$|u_n(x_n, y_n) - u(x_n, y_n)| \geq \delta > 0,$$

where $(x_n, y_n) \in \overline{\Omega}$ satisfies $|x_n| \to \infty$ and $y_n \to y_0$. Define

$$v_n(x,y) = u_n(x + x_n, y) - u(x + x_n, y).$$

Extracting another subsequence, we can assume $v_n \to v$ in $C_{\mathrm{loc}}^2(\overline{\Omega})$, where $v \in C_{\mathrm{b}}^{2+\beta}(\overline{\Omega})$, and also that $L_n v_n \to \widetilde{L} v$ in $C_{\mathrm{loc}}^0(\overline{\Omega}) \times C_{\mathrm{loc}}^1(\Gamma_1)$. Since $L_n v_n = f_n - f \to 0$ in Y_{b}, we must have $\widetilde{L} v = 0$. But

$$|v(0, y_0)| = \lim_{n \to \infty} |u_n(x_n, y_0) - u(x_n, y_0)| \geq \delta,$$

so $v \not\equiv 0$, a contradiction. ◻

A.4. Weighted Hölder spaces. For unbounded domains Ω the inclusion of $C_{\mathrm{b}}^{k+\beta}(\overline{\Omega})$ into $C_{\mathrm{b}}^{\ell+\gamma}(\overline{\Omega})$ with $\ell + \gamma < k + \beta$ is no longer compact. As a replacement, we will use the following elementary lemma.

LEMMA A.8. *Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain and set $\Omega_R = \Omega \cap \{|x| > R\}$. If a sequence $u_n \in C_{\mathrm{b}}^{k+\beta}(\overline{\Omega})$ has $u_n \to u$ in $C_{\mathrm{loc}}^{k+\beta}(\overline{\Omega})$ and satisfies the equidecay condition*

$$\lim_{R \to \infty} \sup_n |u_n|_{k+\beta;\Omega_R} = 0, \tag{A.10}$$

then $u_n \to u \in C_{\mathrm{b}}^{k+\beta}(\overline{\Omega})$. In particular, suppose that $\ell + \gamma > k + \beta$ and that $u_n \in C_{\mathrm{b}}^{\ell+\gamma}(\overline{\Omega})$ is a bounded sequence satisfying (A.10). *Then u_n has a subsequence converging in $C_{\mathrm{b}}^{k+\beta}(\overline{\Omega})$.*

In light of Lemma A.8, it will be convenient to work in function spaces where the norm controls the rate of decay at infinity. Since products of functions decaying at a certain rate will decay even faster, these spaces are especially useful for nonlinear problems.

Let $\sigma\colon \Omega \to (0,\infty)$ be a strictly positive smooth function. We define the weighted Hölder spaces

$$C^{k+\beta}_\sigma(\overline{\Omega}) = \{u \in C^{k+\beta}(\overline{\Omega}) : |\sigma u|_{k+\beta} < \infty\}.$$

An obvious feature of this definition is that $u \mapsto \sigma u$ is an isometric isomorphism $C^{k+\beta}_\sigma(\overline{\Omega}) \to C^{k+\beta}_{\mathrm{b}}(\overline{\Omega})$. The weight functions σ we will consider will usually satisfy

$$\lim_{|x|\to\infty} \sigma = \infty, \qquad \lim_{|x|\to\infty} \frac{D^\alpha \sigma}{\sigma} = 0 \quad \text{for all multi-indices } \alpha \neq 0. \tag{A.11}$$

The first part of (A.11) guarantees that $C^{k+\beta}_\sigma(\overline{\Omega}) \subsetneq C^{k+\beta}_{\mathrm{b}}(\overline{\Omega})$ is a space of functions vanishing as $|x| \to \infty$. The second part guarantees that σ grows more slowly than any exponential $Ce^{k|x|}$. Many of the results of this section only require the weaker hypothesis

$$\sup_x \frac{|D^\alpha \sigma|}{\sigma} < \infty \quad \text{for all multi-indices } \alpha \neq 0. \tag{A.12}$$

To understand the role of the assumptions (A.11) and (A.12), consider a bounded linear operator $A\colon C^{k+\beta}_{\mathrm{b}}(\overline{\Omega}) \to C^{\ell+\beta}_{\mathrm{b}}(\overline{\Omega})$. Questions about A as an operator $C^{k+\beta}_\sigma(\overline{\Omega}) \to C^{\ell+\beta}_\sigma(\overline{\Omega})$ are easily translated into questions about the conjugated operator $A_\sigma(u) = \sigma A(\sigma^{-1}u)$ as a map $C^{k+\beta}_{\mathrm{b}}(\overline{\Omega}) \to C^{\ell+\beta}_{\mathrm{b}}(\overline{\Omega})$. For the operators A we will now consider, $A - A_\sigma$ is bounded when σ satisfies (A.12) and compact when σ satisfies (A.11).

Let $\Omega = \mathbb{R}^n \times (0,1)$, and L, A, B be as in (A.2). The conjugated operators $A_\sigma(u) = \sigma A(\sigma^{-1}u)$ and $B_\sigma u = \sigma B(\sigma^{-1}u)$ are given by

$$\begin{aligned} A_\sigma u &= Au - 2a^{ij}\frac{D_j\sigma}{\sigma}D_i u - \left(a^{ij}\frac{D_{ij}\sigma}{\sigma} - 2a^{ij}\frac{D_i\sigma D_j\sigma}{\sigma^2} + b^i\frac{D_i\sigma}{\sigma}\right)u,\\ B_\sigma u &= Bu - \gamma^i\frac{D^i\sigma}{\sigma}u. \end{aligned} \tag{A.13}$$

Notice that the highest order coefficients of A_σ and B_σ are the same as those for A and B. Moreover, if (A.12) holds, then the coefficients of A_σ are $C^\beta_{\mathrm{b}}(\overline{\Omega})$ and those of B_σ are $C^{1+\beta}_{\mathrm{b}}(\Gamma_1)$. This allows us to easily prove the following lemma.

Lemma A.9. *Suppose that $a^{ij}, b^i, c \in C^\beta_{\mathrm{b}}(\overline{\Omega})$ and $\alpha, \gamma^i \in C^{1+\beta}_{\mathrm{b}}(\Gamma_1)$, and that σ satisfies* (A.12). *If $u \in C^0_\sigma(\overline{\Omega}) \cap C^{2+\beta}(\overline{\Omega})$ solves* (A.1) *with $f \in C^\beta_\sigma(\overline{\Omega})$ and $g \in C^{1+\beta}_\sigma(\overline{\Omega})$, then $u \in C^{2+\beta}_\sigma(\overline{\Omega})$ and u satisfies the Schauder estimate*

$$|\sigma u|_{2+\beta} \le C(|\sigma f|_\beta + |\sigma g|_{1+\beta} + |\sigma u|_0), \tag{A.14}$$

where the constant C depends only on σ, the ellipticity and obliqueness constants, and the stated norms of the coefficients of A, B.

Proof. Simply apply Lemma A.1 with A, B replaced by A_σ, B_σ and u, f, g replaced by $\sigma f, \sigma g, \sigma u$. ∎

Similarly, when (A.11) holds, we can show properness between weighted spaces. Define

$$X_\sigma = \{u \in C^{2+\beta}_\sigma(\overline{\Omega}) : u|_{\Gamma_0} \equiv 0\}, \qquad Y_\sigma = C^\beta_\sigma(\overline{\Omega}) \times C^{k+\beta}_\sigma(\Gamma_1). \tag{A.15}$$

LEMMA A.10. *Suppose that $L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is semi-Fredholm with index $\nu < \infty$ and that σ satisfies* (A.11). *Then L is also semi-Fredholm with index ν as a map $X_\sigma \to Y_\sigma$.*

Proof. Since the Fredholm index of $L\colon X_\sigma \to Y_\sigma$ is the same as the index of the conjugated operator $L_\sigma = (A_\sigma, B_\sigma)\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$, it suffices to show that $L_\sigma - L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is compact. Writing (A.13) as

$$A_\sigma u - Au = b^i_\sigma D_i u + c^i_\sigma u, \qquad B_\sigma u - Bu = \alpha_\sigma u,$$

we note that, thanks to (A.11), the coefficients $b^i_\sigma, c_\sigma, \alpha_\sigma$ satisfy

$$|b^i_\sigma|_{\beta;\Omega_R},\ |c^i_\sigma|_{\beta;\Omega_R},\ |\alpha_\sigma|_{1+\beta;\Omega_R} \to 0 \qquad \text{as } R \to \infty,$$

where $\Omega_R = \{(x,y) \in \Omega : |x| > R\}$. Applying Lemma A.8, we conclude that $L - L_\sigma$ is compact. $\square$

Finally, when (A.11) holds, invertibility in unweighted spaces implies invertibility with weights.

LEMMA A.11. *Suppose that $L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is invertible and that σ satisfies* (A.11). *Then L is also invertible as a map $X_\sigma \to Y_\sigma$.*

Proof. Since $L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$ is invertible, it is Fredholm with index 0, so by Lemma A.10, $L\colon X_\sigma \to Y_\sigma$ is also Fredholm with index 0. It therefore suffices to show that it has trivial kernel. But $X_\sigma \subset X$, so the kernel of $L\colon X_\sigma \to Y_\sigma$ is contained in the kernel of $L\colon X_{\mathrm{b}} \to Y_{\mathrm{b}}$, which is trivial. $\square$

A.5. Properness of nonlinear elliptic operators. In this section we will prove local properness for nonlinear elliptic operators in weighted Hölder spaces. Unlike in Appendices A.1–A.3, the weight function σ will play a central role in the argument. We note that sections 2.4 and 3.6 of Chapter 11 in [47] give an example of a nonlinear elliptic operator between unweighted spaces which fails to be locally proper even though its linearized operators are. There are approaches to local properness in unbounded domains that do not involve weights, see [40], but they do not apply directly to our problem because of its fully nonlinear boundary condition.

Let $\Omega = \mathbb{R} \times (0,1)$, and fix a weight function σ satisfying (A.11). Defining X_σ and Y_σ as in (A.15), let $\mathscr{D} \subset X_\sigma$ be a closed, convex set. We require $\mathscr{D}$ to be the closure of some open subset of X_σ. We also assume that $\mathscr{D}$ has the following property: if u_n is a sequence in $\mathscr{D}$ with $u_n \to u$ in $C^2_{\mathrm{loc}}(\overline{\Omega})$, then $u \in \mathscr{D}$. This condition is easily verified for domains $\mathscr{D}$ defined by pointwise inequalities.

Define a C^2 nonlinear mapping $\mathscr{F}\colon \mathscr{D} \to Y_\sigma$ by

$$\mathscr{F}_1(u)(z) = F_1(z, u, Du, D^2 u), \qquad \mathscr{F}_2(u)(z) = F_2(z, u, Du).$$

We write $F_1 = F_1(z,\eta)$ and $F_2 = F_2(z,\zeta)$, where $\eta \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ and $\zeta \in \mathbb{R} \times \mathbb{R}^2$. We suppose that F_1 is smooth in η and Hölder continuous in z with uniform bounds

$$\sup_{|\eta| \le M} \max_{|\beta| \le 2} \|D^\beta_\eta F_1(\,\cdot\,,\eta)\|_{C^\beta(\Omega)} < \infty, \qquad \sup_{z \in \Omega} \max_{|\beta| \le 2} \|D^\beta_\eta F_1(z,\,\cdot\,)\|_{C^{0,1}(\{|\eta|<M\})} < \infty$$

for any $M > 0$. Similarly we suppose that F_2 is smooth in ζ and Hölder continuous in z with

$$\sup_{|\zeta| \le M} \max_{|\beta| \le 2} \|D^\beta_\zeta F_2(\,\cdot\,,\zeta)\|_{C^\alpha(\Gamma_1)} < \infty, \qquad \sup_{z \in \Gamma_1} \max_{|\beta| \le 2} \|D^\beta_\zeta F_2(z,\,\cdot\,)\|_{C^{0,1}(\{|\zeta|<M\})} < \infty.$$

Finally, we assume that the Fréchet derivative $\mathscr{F}_u(u)\colon X_\sigma \to Y_\sigma$ is locally proper for each $u \in \mathscr{D}$. Note that if $\mathscr{F}_u(0)$ has a limiting problem, then this will also be a limiting problem for $\mathscr{F}_u(u)$ whenever $u \in X_\sigma$.

Lemma A.12. *Under the above assumptions, $\mathscr{F}\colon \mathscr{D} \to Y_\sigma$ is proper.*

Proof. By intersecting $\mathscr{D}$ with a closed ball in X_σ, it suffices to consider the case where $\mathscr{D}$ is bounded. Let u_n be a sequence in $\mathscr{D}$ with $\mathscr{F}(u_n) \to f = (f_1, f_2)$ in Y_σ. We need to show that u_n has a convergent subsequence. As usual, we can extract a subsequence so that $u_n \to u$ in $C^2_{\mathrm{loc}}(\overline{\Omega})$, where $u \in X_\sigma$. By our hypothesis on $\mathscr{D}$, $u \in \mathscr{D}$ and $\mathscr{F}(u) = f$. Taylor expanding in u, we write

$$\mathscr{F}_u(u)[u_n - u] = \mathscr{F}(u_n) - \mathscr{F}(u) + \mathscr{R}(u, u_n),$$

where we think of $\mathscr{R}(u, u_n)$ as a remainder term. By the local properness of $\mathscr{F}_u(u)$, it is enough to show $\mathscr{R}(u, u_n) \to 0$ in Y_σ. Let $v = (u, Du, D^2u)$ and $v_n = (u_n, Du_n, D^2u_n)$. Then

$$\begin{aligned}\mathscr{R}_1(u, u_n)(z) &= \int_0^1 (1-s) F_{1vv}(z, v + s(v_n - v))(v_n - v, v_n - v)\, ds \\ &=: R_{1n}(z)(v_n - v, v_n - v),\end{aligned}$$

where $R_{1n}(z)$ is a quadratic form. By our assumptions on F_1 and the boundedness of $\mathscr{D}$, the coefficients of R_{1n} are bounded in $C^\beta_{\mathrm{b}}(\overline{\Omega})$, uniformly in n. Thus, for any $U \subset \Omega$,

(A.16)
$$|\sigma \mathscr{R}_1(v, v_n)|_{\beta;U} \le C|\sigma^{-1}|_{\beta;U} |\sigma v_n - \sigma v|_{\beta;U} |\sigma v_n - \sigma v|_{0;U} \le C|\sigma^{-1}|_{\beta;U} |\sigma v_n - \sigma v|_{0;U},$$

where C is independent of U. Since $v_n \to v_0$ in $C^0_{\mathrm{loc}}(\overline{\Omega})$, (A.16) gives $\sigma\mathscr{R}_1(v, v_n) \to 0$ in $C^\beta_{\mathrm{loc}}(\overline{\Omega})$. On the other hand, setting $\Omega_r = \{(x, y) \in \Omega : |x| > r\}$, (A.16) gives $|\sigma\mathscr{R}_1(v, v_n)|_{\beta;\Omega_r} \le C|\sigma^{-1}|_{\beta;\Omega_r} \to 0$ as $r \to 0$, uniformly in n. Thus Lemma A.8 implies $|\sigma\mathscr{R}_1(v, v_n)|_\beta \to 0$. Arguing similarly for $\mathscr{R}_2$ we find $\mathscr{R}(u, u_n) \to 0$ in Y_σ as desired. ∎

This result is easily extended to the case where $\mathscr{F}$ depends smoothly on a parameter $\lambda \in [0, 1]$, provided the bounds on F_1, F_2 and their derivatives are satisfied uniformly in λ.

A.6. Problems with symmetry. Let $X_{\mathrm{b}}, X_\sigma, Y_{\mathrm{b}}, Y_\sigma$ and $L = (A, B)$ be as in Appendices A.1 and A.4, and define

$$Ru(x_1, x_2, \dots, x_n, y) = u(-x_1, x_2, \dots, x_n, y).$$

Assume that σ, A, B have the symmetry $R\sigma = \sigma$, $ARu = RAu$, and $BRu = RBu$, and set $X^{\mathrm{e}}_{\mathrm{b}} = \{u \in X : Ru = u\}$ and so on. The following lemma is straightforward.

Lemma A.13. *Under the above assumptions, the results in Appendices* A.1, A.2, A.4, *and* A.5 *remain valid if we replace X_{b} by $X^{\mathrm{e}}_{\mathrm{b}}$, Y_σ by Y^{e}_σ, and so on. As for Lemma* A.7, *suppose that the homogeneous limiting problem $\widetilde{L}u = 0$ has no nontrivial solutions $u \not\equiv 0$ in X_{b}. Then $L^{\mathrm{e}}\colon X^{\mathrm{e}}_{\mathrm{b}} \to Y^{\mathrm{e}}_{\mathrm{b}}$ is locally proper.*

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